Math 225 Midterm 1

Spring 2024

Feb 9

SOLUTIONS

1. (20 points) Examine the propositions below and determine whether they are true or false. You do not need to provide an explanation, just mark your choice. (Each is 4 points)

True — False A is a 3×3 matrix such that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then A does not have an inverse.

True — **False** If A and B are $n \times n$ matrices, and A is invertible, then AB is invertible.

True — **False** Suppose A is a 3×5 matrix. Then $rank(A) = 3$.

True — **False** A homogeneous system of equations can be inconsistent.

True — False For every positive integer n, $\text{adj}(I_n) = I_n$.

- Since $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no unique solution, by IMT, A is not invertible.
- If $B = 0$, for example, we would have $AB = 0$ which is not invertible.
- If A is the 3×5 zero matrix, then its rank would be zero.
- Trivial solution is always a solution for homogeneous systems.
- By adjoint formula for inverses, we get

$$I_n = I_n^{-1} = \frac{1}{\det(I_n)} \text{adj}(I_n) = \text{adj}(I_n).$$

2. Consider the system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 5y + z &= 7 \\ x + y + (k-5)z &= l - 3 \end{aligned}$$

(a) (3 points) Write the augmented matrix of the system.

$$\begin{bmatrix} 1 & 2 & -1 & \mid & 3 \\ 2 & 5 & 1 & \mid & 7 \\ 1 & 1 & k-5 & \mid & l-3 \end{bmatrix}$$

(b) (5 points) Reduce the augmented matrix to row-echelon form .

After applying $A_{12}(-2)$, $A_{13}(-1)$, and $A_{23}(1)$, we get $\begin{bmatrix} 1 & 2 & -1 & \mid & 3 \\ 0 & 1 & 3 & \mid & 1 \\ 0 & 0 & k-1 & \mid & l-5 \end{bmatrix}$. This is enough to answer the other parts, but if you want to get the actual row-echelon form, we should assume $k - 1 \neq 0$ and apply $M_3(\frac{1}{k-1})$ to get $\begin{bmatrix} 1 & 2 & -1 & \mid & 3 \\ 0 & 1 & 3 & \mid & 1 \\ 0 & 0 & 1 & \mid & \frac{l-5}{k-1} \end{bmatrix}$.

(c) (4 points) Find all values of k and l (if any) where the system has no solutions.

If $k = 1$ and $l \neq 5$, then $rank(A) = 2$ but $rank(A^{\#}) = 3$. There is no solution in this case.

(d) (4 points) Find all values of k and l (if any) where the system has a unique solution.

If $k \neq 1$ and l is any number, then $rank(A) = rank(A^{\#}) = 3$. There is a unique solution in this case.

(e) (4 points) Find all values of k and l (if any) where the system has an infinite number of solutions.

If $k = 1$ and $l = 5$ is any number, then $rank(A) = rank(A^{\#}) = 2 < 3$. There are infinitely many solutions in this case.

3. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & a & 2 \end{bmatrix}$.

(a) (7 points) Using the cofactor expansion along the first column, find $\det(A)$.

$$\begin{aligned} \det(A) &= C_{11} + C_{21} \\ &= \det\begin{pmatrix} 0 & -1 \\ a & 2 \end{pmatrix} - \det\begin{pmatrix} -1 & 0 \\ a & 2 \end{pmatrix} \\ &= a + 2 \end{aligned}$$

(b) (3 points) For which values of a, A is invertible?

A is invertible if and only if $\det(A) \neq 0$. By part (a), we get A is invertible for $a \neq -2$.

(c) (10 points) Use Gauss-Jordan method to find the inverse of A when $a = -1$.

Apply $A_{12}(-1)$, $A_{23}(1)$, $A_{32}(1)$, and $A_{21}(1)$ on $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$ to get

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$. Thus, we have $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

4. (20 points) Use Cramer's rule to solve the system

$$\begin{aligned} x - z &= -3 \\ 2x + y &= 2 \\ -2y - z &= -1 \end{aligned}$$

$$x = \frac{\det\begin{bmatrix} -3 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix}}{\det\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}} = \frac{6}{3} = 2$$

$$y = \frac{\det\begin{bmatrix} 1 & -3 & -1 \\ 2 & 2 & 0 \\ 0 & -1 & -1 \end{bmatrix}}{\det\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}} = \frac{-6}{3} = -2$$

$$z = \frac{\det\begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & 2 \\ 0 & -2 & -1 \end{bmatrix}}{\det\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix}} = \frac{15}{3} = 5$$

Thus, $(2, -2, 5)$ is the solution for the system.

5. (20 points) Consider the following matrices, pair them according the their determinants. In other words, if the determinants of two matrices in the list are equal, them indicate that pair. Each right pair is worth 4 points.

$$A_1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad B_1 = \begin{bmatrix} -a & -b & -c \\ 4d & 4e & 4f \\ g & h & i \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{bmatrix} \qquad B_2 = \begin{bmatrix} -3a & d & 4g \\ -3b & e & 4h \\ -3c & f & 4i \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a & c & b \\ d & f & e \\ g & i & h \end{bmatrix} \qquad B_3 = \begin{bmatrix} a & b & c \\ 3a+d & 3b+e & 3c+f \\ g & h & i \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ 6g & 6h & 6i \end{bmatrix} \qquad B_4 = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$$

$$A_5 = \begin{bmatrix} b & a & 4c \\ e & d & 4f \\ h & g & 4i \end{bmatrix} \qquad B_5 = \begin{bmatrix} a & b & c \\ d & e & f \\ 8g & 8h & 8i \end{bmatrix}$$

Pairs are $A_1 - B_3$, $A_2 - B_5$, $A_3 - B_4$, $A_4 - B_2$, and $A_5 - B_1$.