1. (20 points) Examine the propositions below and determine whether they are true or false. You do not need to provide an explanation, just mark your choice. (Each is 4 points)

True — False ✓ The set of positive real numbers, with the usual operations of addition and scalar multiplication, forms a vector space.

True — False ✓ Five vectors in $P_3(\mathbb{R})$ must be linearly dependent.

True — False ✓ A change-of-basis matrix is always invertible.

True — False ✓ An $n \times n$ matrix A for which $nullspace(A) = colspace(A)$ cannot be invertible.

True — False ✓ If $T : P_3(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$ is a linear transformation and $Ran(T)$ is four-dimensional, then T is one-to-one but not onto.

$4 + \dim(\ker(T)) = 4$

2. Prove the following statements

(a) (10 points) Let $T : V \to W$ be a linear transformation, and assume that V and W are both finite dimensional. If T is onto, then

$$dim(V) \geq dim(W).$$

10

If T is onto, then for each $w \in W$ there is a $v \in V$ such that $T(v) = w$. In other words, $\dim(Ran(T)) = \dim(W)$ and $Ran(T) = W$.

By GRN, $\dim(Ker(T)) + \dim(Ran(T)) = \dim(V)$

$\dim(Ker(T)) + \dim(W) = \dim(V)$

$\dim(Ker(T)) \geq 0$ so,

$\dim(W) \leq \dim(V)$ //

(b) (10 points) Let $T : V \to W$ be a linear transformation. Then

$$Ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

10

is a subspace of ~~W~~. V

Let $v, u \in Ker(T)$ and c be a scalar.

$T(v+u) = T(v) + T(u) = 0 + 0 = 0$

$\therefore v + u \in Ker(T)$

$T(cv) = cT(v) = c(0) = 0$

$\therefore cv \in Ker(T)$

So $Ker(T)$ is a subspace of V.

3. Consider the following set of polynomials

$$\begin{aligned} S_1 &= \{x - x^2, 2x - x^2, -1 + 2x - x^2\} \\ S_2 &= \{3 + x + x^2, 1 - x^2\} \\ S_3 &= \{1, 2x, x + 5x^2, 1 + x + x^2\} \end{aligned}$$

(a) (5 points) Which one can be a basis for $P_2(\mathbb{R})$?

S_1 can be a basis.

5

(b) (10 points) For your guess in part (a), use Wronskian method to show the set is linearly independent.

10

$$W[x-x^2, 2x-x^2, -1+2x-x^2](x) = \det\left(\begin{bmatrix} x-x^2 & 2x-x^2 & -1+2x-x^2 \\ 1-2x & 2-2x & 2-2x \\ -2 & -2 & -2 \end{bmatrix}\right)$$

$$W[x-x^2, 2x-x^2, -1+2x-x^2](1) = \det\left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix}\right) \begin{matrix} 0 & 1 \\ -1 & 0 \\ -2 & -2 \end{matrix}$$

$= -2 \neq 0$ $\therefore S_1$ is linearly independent.

(c) (5 points) Why does part (b) finish to prove that your guess is indeed a basis? Explain.

5

Since $\dim(P_2(\mathbb{R})) = 3$, any set of 3 linearly independent vectors in $P_2(\mathbb{R})$ is a basis. Since S_1 has 3 vectors and they are linearly independent, S_1 is a basis for $P_2(\mathbb{R})$.

4. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a, b) = (a + b, a - b, 2a + 5b).$$

(a) (7 points) Find the matrix representation of T.

$T(1,0) = (1,1,2)$

$T(0,1) = (1,-1,5)$

$\therefore T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 5 \end{bmatrix}$.

7

(b) (8 points) Find $Ker(T)$ and $Ran(T)$.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 5 \end{bmatrix} \xrightarrow[A_{13}(-2)]{A_{12}(-1)} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 3 \end{bmatrix} \xrightarrow[M_3(\frac{1}{3})]{M_2(-\frac{1}{2})} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow[A_{23}(-1)]{A_{21}(-1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

8

$Ker(T) = \text{nullspace}(A) = \{\vec{0}\}$

$Ran(T) = \text{colspace}(A) = \text{span}\{(1,1,2), (1,-1,5)\}$

(c) (5 points) Determine whether T is one-to-one and/or onto?

T is one-to-one because $Ker(T) = \{\vec{0}\}$. T is not onto because $Ran(T) \neq \mathbb{R}^3$.

5

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$.

(a) (4 points) Define *rowspace*(A) and *colspace*(A).

rowspace(A) is the span of the rows of A.

colspace(A) is the span of the columns of A.

✓ 4

(b) (6 points) Reduce A into the reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix} \xrightarrow[A_{13}(-3)]{A_{12}(-1)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow[M_2(\frac{1}{2})]{A_{23}(-2)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

✓ 6

(c) (10 points) Find the bases for *rowspace*(A) and *colspace*(A).

$\{(1,0,1),(0,1,0)\}$ is a basis for rowspace(A). ✓

$\{(1,1,3),(0,2,4)\}$ is a basis for colspace(A).

10

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.