Homework 1 Solutions

1. Write $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}$ and $B = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$. Then

   (a) The second column of AB is obtained by $\begin{bmatrix} r_1 \cdot c_2 \\ r_2 \cdot c_2 \\ r_3 \cdot c_2 \\ r_4 \cdot c_2 \\ r_5 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}$.

   (b) The third row AB is obtained by $\begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & r_1 \cdot c_3 & r_1 \cdot c_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 2 \end{bmatrix}$.

   (c) The entry at fifth row and fourth column of the product AB is $r_5 \cdot c_4 = 3$.

2. Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$. Then we have

$$\begin{aligned} tr(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right) \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ik} b_{ki} \right) \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right) \\ &= \sum_{k=1}^{n} (BA)_{kk} = tr(BA) \end{aligned}$$

   E.g. $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

   Then $AB = \begin{bmatrix} 3 & 5 & 2 \\ 4 & 6 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ and $BA = \begin{bmatrix} 4 & 1 & 4 \\ 2 & 1 & 4 \\ 3 & 2 & 6 \end{bmatrix}$. $AB \neq BA$ but $tr(AB) = 11 = tr(BA)$.

3. Let $A(t) = \begin{bmatrix} t(t-1) & 0 & 7t(t-1) \\ t^2(t-5) & 2 & t^3(2t-1) \end{bmatrix}$ and $(t) = \begin{bmatrix} 3t(t-1) & 0 & 10t(t-1) \\ t^2(t-5) & 2 & t^3(2t-1) \end{bmatrix}$.

   Then $A(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, $A(1) = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix}$, $B(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, $B(1) = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix}$.

So $A(0) = B(0)$ and $A(1) = B(1)$.

However, they are distinct because $A(2) = \begin{bmatrix} 2 & 0 & 14 \\ -12 & 2 & 24 \end{bmatrix}$ but $B(2) = \begin{bmatrix} 6 & 0 & 20 \\ -12 & 2 & 24 \end{bmatrix}$.

4. (a) We need to show that $B^T = B$.

$$B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A^T + A) = B.$$

Also, we will show that $C^T = -C$.

$$C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C.$$

(b) We have

$$B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}(A + A^T + A - A^T) = \frac{1}{2}(2A) = A.$$

5. (a) Let's write the product $A_{m\times n}B_{?\times ?}C_{r\times s}$. We should have the number of the columns of $A =$ the number of the rows of B, and the number of the columns of $B =$ the number of the rows of C. Therefore, B is an $n \times r$ matrix.

(b) Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$. Note that AB is an $m \times r$ matrix, and ABC is an $m \times s$ matrix. For $1 \le i \le m$ and $1 \le j \le s$, we can write

$$(ABC)_{ij} = \sum_{k=1}^{r} (AB)_{ik} c_{kj} = \sum_{k=1}^{r} \left(\sum_{l=1}^{n} a_{il} b_{lk} \right) c_{kj} = \sum_{k=1}^{r} \sum_{l=1}^{n} a_{il} b_{lk} c_{kj}.$$