Homework 2 Solutions

1. (a) $A = \begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ Apply P_{13}, $A_{12}(-2)$, $A_{13}(-3)$, $M_2(-1)$, $A_{23}(-1)$, and $M_3(\frac{1}{4})$.

   Then row-echelon form of A is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, so $rank(A) = 3$.

   (b) $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -2 & 0 & 7 \\ 2 & -1 & 2 & 4 \\ 4 & -2 & 3 & 8 \end{bmatrix}$ Apply $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-4)$, $A_{32}(-1)$, $A_{34}(-2)$, P_{23},

   $A_{34}(-1)$, $M_3(-1)$, $M_4(-1)$. Then row-echelon form of A is $\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

   so $rank(A) = 4$.

2. Consider the augmented matrix of the system:

$$\begin{bmatrix} 1 & -1 & 0 & -1 & | & 2 \\ 2 & 1 & 3 & 7 & | & 2 \\ 3 & -2 & 1 & 0 & | & 4 \end{bmatrix}.$$

   Apply $A_{12}(-2)$, $A_{13}(-3)$, $M_2(\frac{1}{3})$, $A_{23}(-1)$, then we get

$$\begin{bmatrix} 1 & -1 & 0 & -1 & | & 2 \\ 0 & 1 & 1 & 3 & | & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & | & -\frac{4}{3} \end{bmatrix}.$$

   Since $rank(A) = 2 < 3 = rank(A^{\#})$, there is no solution for the system.

3. (a) Consider the augmented matrix of the system

$$\begin{bmatrix} 1 & 0 & 3 & | & 0 \\ 3 & -1 & 7 & | & 0 \\ 2 & 1 & 8 & | & 0 \\ 1 & 1 & 5 & | & 0 \\ -1 & 1 & -1 & | & 0 \end{bmatrix}.$$

   Apply $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-1)$, $A_{15}(1)$, $A_{23}(1)$, $A_{24}(1)$, $A_{25}(1)$, $M_2(-1)$, then we get

$$\begin{bmatrix} 1 & 0 & 3 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Since $rank(A) = rank(A^{\#}) = 2 < 3$, there are infinitely many solutions. Take z as a free variable s. Then, the solution is given by the set $\{(-3s, -2s, s)\}$.

(b) Consider the augmented matrix of the system

$$\begin{bmatrix} 2+i & i & 3-2i & | & 0 \\ i & 1-i & 4+3i & | & 0 \\ 3-i & 1+i & 1+5i & | & 0 \end{bmatrix}.$$

Apply $A_{12}(-\frac{i(2-i)}{5})$, $A_{13}(-\frac{(2-i)(3-i)}{5})$, $M_1(\frac{1}{2+i})$, $M_2(\frac{5}{7-6i})$, $M_3(\frac{1}{10i})$, then we get

$$\begin{bmatrix} 1 & \frac{i}{2+i} & \frac{3-2i}{2+i} & | & 0 \\ 0 & 1 & \frac{13+11i}{7-6i} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}.$$

Since $rank(A) = rank(A^{\#}) = 3$, there is a unique solution, namely, the trivial solution $(0, 0, 0)$.

4. Consider the augmented matrix of the system

$$\begin{bmatrix} k & 1 & 1 & | & 1 \\ 1 & k & 1 & | & 1 \\ 1 & 1 & k & | & 1 \end{bmatrix}.$$

Apply P_{12}, $A_{12}(-k)$, $A_{13}(-1)$, P_{23}, $A_{23}(-(1+k))$, then we get

$$\begin{bmatrix} 1 & k & 1 & | & 1 \\ 0 & 1-k & k-1 & | & 0 \\ 0 & 0 & (1-k)(2+k) & | & 1-k \end{bmatrix}.$$

(Case 1) If $\mathbf{k = 1}$, then we have $\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$. Since $rank(A) = rank(A^{\#}) = 1 < 3$, **there are infinitely many solutions**.

(Case 2) If $\mathbf{k = -2}$, then we have $\begin{bmatrix} 1 & -2 & 1 & | & 1 \\ 0 & 3 & -3 & | & 0 \\ 0 & 0 & 0 & | & 3 \end{bmatrix}$. Since $rank(A) = 2 < 3 = rank(A^{\#})$, **there is no solution for the system**.

(Case 3) If $\mathbf{k \neq 1, -2}$, we can also apply $M_2(\frac{1}{1-k})$ and $M_3(\frac{1}{(1-k)(2+k)})$, and get $\begin{bmatrix} 1 & k & 1 & | & 1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & \frac{1}{2+k} \end{bmatrix}$.

Since $rank(A) = rank(A^{\#}) = 3$, **there is a unique solution**.

5. Consider the augmented matrix of the system

$$\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 2 & -3 & a & | & b \\ -1 & 4 & -2 & | & 5 \end{bmatrix}.$$

Apply $A_{12}(-2)$, $A_{13}(1)$, P_{23}, $M_2(\frac{1}{6})$, $A_{23}(7)$, then we get

$$\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & -\frac{1}{2} & | & \frac{8}{6} \\ 0 & 0 & a-\frac{3}{2} & | & b+\frac{20}{6} \end{bmatrix}.$$

(Case 1) Suppose a $\neq \mathbf{\frac{3}{2}}$. Then we can apply $M_3(\frac{1}{a-\frac{3}{2}})$ and get $\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & -\frac{1}{2} & | & \frac{8}{6} \\ 0 & 0 & 1 & | & (\frac{1}{a-\frac{3}{2}})(b+\frac{20}{6}) \end{bmatrix}$. Since $rank(A) = rank(A^{\#}) = 3$, whatever b is, **there is a unique solution**.

(Case 2) Suppose **a** $= \mathbf{\frac{3}{2}}$. If **b** $= -\mathbf{\frac{20}{6}}$, then we have $\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & -\frac{1}{2} & | & \frac{8}{6} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$. Since $rank(A) = rank(A^{\#}) = 2 < 3$, **there are infinitely many solutions**.

(Case 3) Suppose **a** $= \mathbf{\frac{3}{2}}$ but **b** $\neq -\mathbf{\frac{20}{6}}$, then we have $\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & -\frac{1}{2} & | & \frac{8}{6} \\ 0 & 0 & 0 & | & b+\frac{20}{6} \neq 0 \end{bmatrix}$. Since $rank(A) = 2 < 3 = rank(A^{\#})$, **there is no solution for the system**.