Homework 3 Solutions

1. (a) First, we need to find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. We apply Gauss-Jordan technique, namely, reduce

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 5 & | & 0 & 1 \end{bmatrix}.$$

Apply $A_{12}(-2)$, $M_2(-1)$, and $A_{21}(-3)$, and get

$$\begin{bmatrix} 1 & 0 & | & -5 & 3 \\ 0 & 1 & | & 2 & -1 \end{bmatrix}.$$

Thus, the inverse is $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$, and the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

(b) First, we need to find the inverse of $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix}$. We apply Gauss-Jordan technique, namely, reduce

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 2 & 4 & -3 & | & 0 & 0 & 1 \end{bmatrix}.$$

Apply $A_{13}(-2)$, $A_{23}(-2)$, $M_3(-1)$, $A_{32}(-1)$, and $A_{21}(-1)$, and get

$$\begin{bmatrix} 1 & 0 & 0 & | & 7 & 5 & -3 \\ 0 & 1 & 0 & | & -2 & -1 & 1 \\ 0 & 0 & 1 & | & 2 & 2 & -1 \end{bmatrix}.$$

Thus, the inverse is $\begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$, and the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

2. We apply Gauss-Jordan technique, namely, reduce

$$\begin{bmatrix} 1 & a & 0 & | & 1 & 0 & 0 \\ 0 & 1 & a & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}.$$

Apply $A_{32}(-a)$ and $A_{21}(-a)$, and get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & a^2 \\ 0 & 1 & 0 & 0 & 1 & -a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus, the inverse is $\begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}$.

3. (a) It is enough to observe that

$$(ABC)(C^{-1}B^{-1}A^{-1}) = (AB)I_n(B^{-1}A^{-1}) = (AB)(B^{-1}A^{-1}) = AI_nA^{-1} = AA^{-1} = I_n.$$

(b) Since A is invertible, A^{-1} exists. So multiply the given equality with A^{-1} from the left, and we get

$$B = I_nB = A^{-1}(AB) = A^{-1}(AC) = I_nC = C.$$

(c) Since A is invertible, A^{-1} exists. So multiply the given equality with A^{-1} from the right, and we get

$$B = BI_n = (BA)A^{-1} = (CA)A^{-1} = CI_n = C.$$

(d) Take $I_n + A + A^2 + A^3 + \ldots + A^{k-1}$ as the inverse. We get

$$\begin{aligned} (I_n - A)(I_n + A + A^2 + A^3 + \ldots + A^{k-1}) &= I_n^2 + I_nA + I_nA^2 + I_nA^3 + \ldots I_nA^{k-1} \\ &\quad -AI_n - A^2 - A^3 - \ldots - A^k \\ &= I_n + A + A^2 + A^3 + \ldots + A^{k-1} \\ &\quad -A - A^2 - A^3 - \ldots - A^k \\ &= I_n - A^k \\ &= I_n \end{aligned}$$

4. (a) Since A is idempotent, we get $A^2 = A$. Then

$$(I_n - A)(I_n - A) = I_n^2 - I_nA - AI_n + A^2 = I_n - A - A + A = I_n - A.$$

Since we get $(I_n - A)^2 = I_n - A$, we conclude that $I_n - A$ is idempotent.

(b) Since A is idempotent, we get $A^2 = A$. Then

$$(2A - I_n)(2A - I_n) = 4A^2 - 2AI_n - I_n2A + I_n = 4A - 2A - 2A + I_n = I_n.$$

Thus, $2A - I_n$ is invertible, and its inverse is again $2A - I_n$.

5. Since A and B are invertible, by IMT, there are elementary matrices $E_1, \ldots, E_m$ and $F_1, \ldots, F_k$ such that $A = E_1 \ldots E_m$ and $B = F_1 \ldots F_k$. Then we obtain

$$AB = E_1 \ldots E_mF_1 \ldots F_k.$$

Since AB is written as a product of elementary matrices, we have (by IMT) AB is invertible.