Homework 4 Solutions

1. (a) First, we reduce $A = \begin{bmatrix} 3 & 7 & 1 \\ 5 & 9 & -6 \\ 2 & 1 & 3 \end{bmatrix}$. Apply $A_{31}(-1)$, $A_{32}(-2)$, $A_{12}(-1)$, $A_{13}(-2)$, and $A_{23}(11)$, then we get $\begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -10 \\ 0 & 0 & -103 \end{bmatrix}$. This is upper triangular, and its determinant is $1 * 1 * (-103) = -103$. Since we only applied addition operation, by **P3** we have $\det(\mathbf{A}) = \mathbf{-103}$.

   (b) First, we reduce $B = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & 2 \\ 4 & 1 & 4 & 3 \\ 5 & 2 & 5 & 3 \end{bmatrix}$. Apply $A_{12}(-3/2)$, $A_{13}(-2)$, $A_{14}(-5/2)$, $A_{23}(-2/3)$, $A_{24}(-1/3)$, and $A_{34}(4)$ then we get $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & -3/2 & -7/2 & -11/2 \\ 0 & 0 & 1/3 & -10/3 \\ 0 & 0 & 0 & -21 \end{bmatrix}$. This is upper triangular, and its determinant is $2 * (-3/2) * (1/3) * (-21) = 21$. Since we only applied addition operation, by **P3** we have $\det(\mathbf{B}) = \mathbf{21}$.

2. We know that the system has infinite number of solutions if and only if the determinant of the coefficient matrix is 0. Use the cofactor expansion at third column:

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & k \\ 2 & -k & 1 \\ 3 & 6 & 1 \end{bmatrix}\right) &= kC_{13} + C_{23} + C_{33} \\ &= k\det\left(\begin{bmatrix} 2 & -k \\ 3 & 6 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 2 \\ 2 & -k \end{bmatrix}\right) \\ &= 3k^2 + 11k - 4 = (\mathbf{3k-1})(\mathbf{k+4}) \end{aligned}$$

   Therefore, the system has infinitely many solutions iff $\mathbf{k = 1/3}$ or $\mathbf{k = -4}$.

3. Let A and B be 4×4 matrices such that $\det(A) = 2$ and $\det(B) = -6$.

   (a) $\det(AB^T) = \det(A)\det(B^T) = \det(A)\det(B) = -12$.

   (b) $\det(A^{-1}(5B)) = \det(A^{-1})\det(5B) = \frac{1}{\det(A)}5^4\det(B) = \frac{1}{2}5^4(-6) = -1875$.

   (c) $\det(B^2A^3) = \det(B)^2\det(A)^3 = (-6)^2 2^3 = 288$.

   (d) $\det((A^TB^{-1})^2) = (\det(A^T)\det(B^{-1}))^2 = (\det(A)\frac{1}{\det(B)})^2 = (-\frac{1}{3})^2 = \frac{1}{9}$.

   (e) $\det(B^{-1}(2A)B^T) = \det(B^{-1})\det(2A)\det(B^T) = \frac{1}{\det(B)}2^4\det(A)\det(B) = 2^4(2) = 32$.

4. We have several expansions:

$$\begin{aligned}
\det\left(\begin{bmatrix}2&0&-1&3&0\\0&3&0&1&2\\0&1&3&0&4\\1&0&1&-1&0\\3&0&2&0&5\end{bmatrix}\right) &= 3C_{22}+C_{32}\\
&= 3\det\left(\begin{bmatrix}2&-1&3&0\\0&3&0&4\\1&1&-1&0\\3&2&0&5\end{bmatrix}\right)-\det\left(\begin{bmatrix}2&-1&3&0\\0&0&1&2\\1&1&-1&0\\3&2&0&5\end{bmatrix}\right)\\
&= 3(3C_{22}+4C_{24})-(C_{23}+2C_{24})\\
&= 3\left(3\det\left(\begin{bmatrix}2&3&0\\1&-1&0\\3&0&5\end{bmatrix}\right)+4\det\left(\begin{bmatrix}2&-1&3\\1&1&-1\\3&2&0\end{bmatrix}\right)\right)\\
&\quad -\left(-\det\left(\begin{bmatrix}2&-1&0\\1&1&0\\3&2&5\end{bmatrix}\right)+2\det\left(\begin{bmatrix}2&-1&3\\1&1&-1\\3&2&0\end{bmatrix}\right)\right)\\
&= 3(3(-25)+4(4))-(-(15)+2(4))=\mathbf{-170}.
\end{aligned}$$

5. Cramer's rule gives that

- $x=\dfrac{\det\left(\begin{bmatrix}-1&1&2\\-1&-1&1\\-5&5&5\end{bmatrix}\right)}{\det\left(\begin{bmatrix}3&1&2\\2&-1&1\\0&5&5\end{bmatrix}\right)}=\frac{-10}{-20}=\frac{1}{2}.$
- $y=\dfrac{\det\left(\begin{bmatrix}3&-1&2\\2&-1&1\\0&-5&5\end{bmatrix}\right)}{\det\left(\begin{bmatrix}3&1&2\\2&-1&1\\0&5&5\end{bmatrix}\right)}=\frac{-10}{-20}=\frac{1}{2}.$
- $z=\dfrac{\det\left(\begin{bmatrix}3&1&-1\\2&-1&-1\\0&5&-5\end{bmatrix}\right)}{\det\left(\begin{bmatrix}3&1&2\\2&-1&1\\0&5&5\end{bmatrix}\right)}=\frac{30}{-20}=-\frac{3}{2}.$

Thus $(\frac{1}{2},\frac{1}{2},-\frac{3}{2})$ is the solution for the system.