Homework 5 Solutions

1. (a) On $\mathbb{R}^2$, define the operations of addition and scalar multiplication as follows:

$$\begin{aligned}(x_1, x_2) \oplus (y_1, y_2) &:= (x_1 - x_2, y_1 - y_2)\\ k \odot (x_1, x_2) &:= (-kx_1, -kx_2)\end{aligned}$$

Which of the conditions for a vector space are satisfied with these operations? Is this a vector space structure?

We need to check ten properties one-by-one

i. **Closure under Vector Addition:** $(x_1, x_2) \oplus (y_1, y_2) = (x_1 - x_2, y_1 - y_2) \in \mathbb{R}^2$. So this condition holds.

ii. **Closure under Scalar Multiplication:** $k \odot (x_1, x_2) = (-kx_1, -kx_2) \in \mathbb{R}^2$. So this condition holds.

iii. **Commutativity of Vector Addition:** We have $(1, 0) \oplus (0, 1) = (1, -1)$ but $(0, 1) \oplus (1, 0) = (-1, 1)$. So this condition does not hold.

iv. **Associativity of Vector Addition:** We have $((1, 0) \oplus (0, 1)) \oplus (0, 0) = (2, 0)$ but $(1, 0) \oplus ((0, 1) \oplus (0, 0)) = (1, -1)$, so this property does not hold.

v. **Existence of Additive Identity:** If there is $(a, b) \in \mathbb{R}^2$ such that for all $(x, y) \in \mathbb{R}^2$ we have

$$(x, y) \oplus (a, b) = (x, y),$$

then we would have $(x - y, a - b) = (x, y)$ which is not always true. So there is no additive identity, namely, the zero vector.

vi. **Existence of Additive Inverses:** Since there is no zero vector, there cannot be additive inverses.

vii. **Identity Element of Scalar Multiplication:** We have $1 \odot (x, y) = (-x, -y)$ which is not always equal to (x, y). So the condition does not hold.

viii. **Distributivity of Scalar Multiplication with respect to Vector Addition:** We have $k \odot ((x_1, x_2) \oplus (y_1, y_2)) = (-k(x_1 - x_2), -k(y_1 - y_2))$ and $(k \odot (x_1, x_2)) \oplus (k \odot (y_1, y_2)) = ((-kx_1) - (-kx_2), (-ky_1) - (-ky_2))$. Since they are equal, the condition holds.

ix. **Distributivity of Scalar Multiplication with respect to Scalar Addition:** We have $(k + l) \odot (x, y) = (-(k + l)x, -(k + l)y)$ but $(k \odot (x, y)) \oplus (l \odot (x, y)) = ((-kx) - (-ky), (-lx) - (-ly))$. Since they are not equal, the condition does not hold.

x. **Compatibility of Scalar Multiplication with Scalar Multiplication:** We have $k \odot (l \odot (x, y)) = ((-k)(-l)x, (-k)(-l)y) = (klx, kly)$, but $(kl) \odot (x, y) = (-klx, -kly)$. Since they are not always equal, the condition does not hold.

In summary, only the conditions 1,2, and 8 are satisfied. $\mathbb{R}^2$ is not a vector space with these operations.

(b) On $M_2(\mathbb{R})$, define the operation of addition by

$$A \oplus B := AB,$$

and use the usual scalar multiplication. Determine which conditions for a vector space are satisfied by $M_2(\mathbb{R})$ with these operations.

We need to check ten properties one-by-one as in the previous part.

(a) Since $A \oplus B = AB \in M_2(\mathbb{R})$, the condition holds.

(b) Since $kA \in M_2(\mathbb{R}^2)$, the condition holds.

(c) We have $A \oplus B = AB$ but $B \oplus A = BA$ Since $AB \neq BA$ in general, the property does not hold.

(d) Since matrix multiplication is associative, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. So the condition holds.

(e) We can take I_2, the identity matrix of size 2×2, as the zero vector because $A \oplus I_2 = AI_2 = A$ for any $A \in M_2(\mathbb{R}^2)$. So the condition holds.

(f) For $A \in M_2(\mathbb{R}^2)$, if we additive inverse $\tilde{A}$ such that $A \oplus \tilde{A} = A\tilde{A} = I_2$, namely, A would be invertible. Since there are noninvertible matrices, the inverse condition does not hold in general.

(g) Since $1A = A$, the condition trivially holds.

(h) Since $k(A \oplus B) = kAB$ but $kA \oplus kB = k^2AB$, the condition does not hold.

(i) Since $(k+l)A = kA + lA$ but $kA \oplus lA = klA$, and they are not the same in general, the property does not hold.

(j) Since $k(lA) = (kl)A$, we have this property.

In summary, only the conditions 1,2,4,5,7 and 10 are satisfied. $M_2(\mathbb{R})$ is not a vector space with these operations.

2. Determine whether given sets S are a subspace of the given vector spaces V

(a) $S = \{(x, y) \big| x^2 - y^2 = 0\}$ and $V = \mathbb{R}^2$.

$(1, 1) \in S$ and $(1, -1) \in S$, but $(1, 1)+(1, -1) = (2, 0) \notin S$. So S is not a subspace of $\mathbb{R}^2$.

(b) $S = \{A \in M_n(\mathbb{R}) \big| tr(A) = 0\}$ and $V = M_n(\mathbb{R})$.

Since $tr(A + B) = tr(A) + tr(B)$ and $tr(kA) = k(tr(A))$, the subset S is closed under addition and scalar multiplication. So S is a subspace of $M_n(\mathbb{R})$.

3. Prove that the space of polynomials of degree n or less, namely P_n, is a subspace of the space of real valued functions $Fun(\mathbb{R}, \mathbb{R})$. Hint: The proof is just one sentence :)

Since P_n is a subset of $Fun(\mathbb{R}, \mathbb{R})$, and P_n is already a vector space with the usual function addition and scalar multiplication, we have P_n is a subspace of $Fun(\mathbb{R}, \mathbb{R})$.

4. Determine whether the given vector **v** is an element of $span\{v_1, v_2\}$.

- $\mathbf{v} = (3, 3, 4)$, $v_1 = (1, -1, 2)$, $v_2 = (2, 1, 3)$.
  We want to determine whether there are scalars a, b such that $av_1 + bv_2 = \mathbf{v}$. This yields a system of equations as
  $$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}.$$
  The reduced row-echelon form of augmented matrix $\begin{bmatrix} 1 & 2 & | & 3 \\ -1 & 1 & | & 3 \\ 2 & 3 & | & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$.
  So we can take $a = -1, b = 2$. And YES!, we have $\mathbf{v} \in span\{v_1, v_2\}$.
- $\mathbf{v} = (5, 3, -6)$, $v_1 = (-1, 1, 2)$, $v_2 = (3, 1, -4)$.
  We want to determine whether there are scalars a, b such that $av_1 + bv_2 = \mathbf{v}$. This yields a system of equations as
  $$\begin{bmatrix} -1 & 3 \\ 1 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}.$$
  The reduced row-echelon form of augmented matrix $\begin{bmatrix} -1 & 3 & | & 5 \\ 1 & 1 & | & 3 \\ 2 & -4 & | & -6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$.
  So we can take $a = 1, b = 2$. And YES!, we have $\mathbf{v} \in span\{v_1, v_2\}$.
- $\mathbf{v} = (1, 1, -2)$, $v_1 = (3, 1, 2)$, $v_2 = (-2, -1, 1)$.
  We want to determine whether there are scalars a, b such that $av_1 + bv_2 = \mathbf{v}$. This yields a system of equations as
  $$\begin{bmatrix} 3 & -2 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$
  The reduced row-echelon form of augmented matrix $\begin{bmatrix} 3 & -2 & | & 1 \\ 1 & -1 & | & 1 \\ 2 & 1 & | & -2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$.
  Since the system has no solution, NO!, we have $\mathbf{v} \notin span\{v_1, v_2\}$.

5. Determine a spanning set for the null space of $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 & 2 \\ 2 & 4 & 6 & -1 \end{bmatrix}$.

   The reduced row echelon form of the matrix A is given by:
   $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $x_3 = t$ (where t is a free parameter), then:

$$x_1 = -t$$
$$x_2 = -t$$
$$x_3 = t$$
$$x_4 = 0$$

Thus, the vector that spans the null space of A can be written in terms of t as:

$$nullspace(A) = \{t(-1, -1, 1, 0) \mid t \in \mathbb{R}\}$$

Therefore, a spanning set for the null space of A is given by:

$$\{(-1, -1, 1, 0)\}.$$