Homework 6 Solutions

1. Determine whether the given set of vectors is linearly dependent or independent in R^n for given n. In the case of linear dependence, find a dependency relationship.

   (a) $\{(2,-1),(3,2),(0,1)\}$, $n=2$.
   Since there are three $\mathbb{R}^2$ vectors, they must be linear dependent. We have

   $$3(2,-1)-2(3,2)-(0,1)=(0,0),$$

   You can find these scalars via Gauss-Jordan elimination, but in this case, it might be easy to guess.

   (b) $\{(1,2,3),(1,-1,2),(1,-4,1)\}$, $n=3$.
   Let $A=\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -4\\ 3 & 2 & 1\end{bmatrix}$. Since $\det(A)=0$, the vectors are linearly dependent. To get the dependency equation, we should solve $A\mathbf{x}=\mathbf{0}$. Reduce the augmented matrix $\begin{bmatrix}1 & 1 & 1 & | & 0\\ 2 & -1 & -4 & | & 0\\ 3 & 2 & 1 & | & 0\end{bmatrix}$. Apply $A_{12}(-2), A_{13}(-3), A_{32}(-3), P_{23}, M_2(-1), A_{21}(-1)$ and get $\begin{bmatrix}1 & 0 & -1 & | & 0\\ 0 & 1 & 2 & | & 0\\ 0 & 0 & 0 & | & 0\end{bmatrix}$. Although there are infinitely many solutions, for the question, we care at least one nontrivial solution. Take $z=1, y=-2, x=1$, so we have

   $$(1,2,3)-2(1,-1,2)+(1,-4,1)=(0,0,0).$$

   If you find the scalars via guess & check, it is also fine. Otherwise, you can always use Gauss-Jordan elimination.

   (c) $\{(1,-1,2,3),(2,-1,2,-1),(-1,1,1,1)\}$, $n=4$.
   Since there are three $\mathbb{R}^4$ vectors, there is no shortcut, namely, we should solve the system directly. Let $A=\begin{bmatrix}1 & 2 & -1\\ -1 & -1 & 1\\ 2 & 2 & 1\\ 3 & -1 & 1\end{bmatrix}$. We need to solve $A\mathbf{x}=\mathbf{0}$. Reduced row echelon form of $\begin{bmatrix}1 & 2 & -1 & | & 0\\ -1 & -1 & 1 & | & 0\\ 2 & 2 & 1 & | & 0\\ 3 & -1 & 1 & | & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 0 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & 0 & 1 & | & 0\\ 0 & 0 & 0 & | & 0\end{bmatrix}$. Thus, there is only trivial solution which means these vectors are linearly independent.

2. Determine all values of the constant k for which the given vectors $(1,1,k)$, $(0,2,k)$, and $(1,k,6)$ are linearly dependent in $\mathbb{R}^3$.

These vectors are linearly dependent if and only if

$$\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & k \\ k & k & 6 \end{bmatrix}\right) = -k^2 - k + 12 = (-k+3)(k+4) = 0.$$

Thus, we have $k = 3, -4$.

3. Determine whether the following statements are true or false. Give explanation for your answers.

   (a) If a set contains the zero vector, it is linearly dependent.
   True because of the next statement. The zero vector is always in $spanS$ for any set of vectors S.

   (b) If $\mathbf{z} \in span\{\mathbf{x}, \mathbf{y}\}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent.
   True. If a, b are scalars such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$, then $\mathbf{z} - a\mathbf{x} - b\mathbf{y} = \mathbf{0}$, so they are dependent.

   (c) If a 2×2 matrix has linearly independent columns, then its columns span $\mathbf{R}^2$.
   True. If $A = [\mathbf{v}_1\ \mathbf{v}_2]$ is 2×2 matrix such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, it means $\det(A) \neq 0$. Therefore, for any $\mathbf{v} \in \mathbb{R}^2$ we have $A\mathbf{x} = \mathbf{v}$ is consistent, so any $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$. This means that $\mathbb{R}^2 = span\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (d) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $\mathbf{x}$ is not a multiple of $\mathbf{y}$, then $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.
   False. Take $\mathbf{y} = (0, 0, 0)$ and $\mathbf{x} = (1, 2, 3)$. Then $\mathbf{x}$ is not a multiple of $\mathbf{y}$, but $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent.

4. Determine whether given set of vectors is linearly independent.

   (a) $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$ in $M_2(\mathbb{R})$.
   Let $x, y, z \in \mathbb{R}$ be such that $xA_1 + yA_2 + zA_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then we get the following equation system

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

   The row-echelon form of the augmented matrix is $\begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$. Since the system has infinitely many solutions, $\{A_1, A_2, A_3\}$ is linearly dependent.

(b) $p_1(x) = 1 - 3x^2$, $p_2(x) = 2x + x^2$, $p_3(x) = 5$ in $P_2(\mathbb{R})$.
Let $a, b, c \in \mathbb{R}$ be such that

$$a(1 - 3x^2) + b(2x + x^2) + c(5) = 0$$

which means

$$(-3a + b)x^2 + (2b)x + (a - 5c) = 0.$$

Then, we can immediately derive that $b = 0$, and then $a = 0$, and so $c = 0$. Therefore, $\{p_1, p_2, p_3\}$ is linearly independent.

5. Use the Wronskian to show that given functions are linearly independent on the given interval I.

(a) $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, $I = (-\infty, \infty)$.
$W[f_1, f_2, f_3](x) = \det\left(\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}\right) = 2 \neq 0$ whatever x is. By Wronskian theorem, $\{f_1, f_2, f_3\}$ is linearly independent.

(b) $f_1(x) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \tan x$, $I = (-\frac{\pi}{2}, \frac{\pi}{2})$.
$W[f_1, f_2, f_3](x) = \det\left(\begin{bmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ -\cos x & -\cos x & 2\sec^2 x \tan x \end{bmatrix}\right) = -\frac{2\sin x}{\cos^3 x} - \tan x.$
Since $W[f_1, f_2, f_3](\frac{\pi}{4}) = -5 \neq 0$, by Wronskian theorem, $\{f_1, f_2, f_3\}$ is linearly independent.