Homework 7 Solutions

1. (a) $n = 1$, $S = \{2 - 5x, 3x, 7 + x\}$
   Since $dim(P_1(\mathbb{R})) = 2$ and S has 3 element, S is linearly dependent, so not a basis.

   (b) $n = 2$, $S = \{-2x + x^2, 1 + 2x + 3x^2, 1 - x^2, 5x + 5x^2\}$
   Since $dim(P_2(\mathbb{R})) = 3$ and S has 4 element, S is linearly dependent, so not a basis.

   (c) $n = 3$, $S = \{1 + x^3, x + x^3, x^2 + x^3\}$
   Since $dim(P_3(\mathbb{R})) = 4$ and S has 3 element, S is not a spanning set, so not a basis.

   (d) $n = 3$, $S = \{1 + x + 2x^2, 2 + x + 3x^2 - x^3, -1 + x + x^2 - 2x^3, 2 - x + x^2 + 2x^3\}$
   Since $dim(P_3(\mathbb{R})) = 4$ and S has 4 element, S can be a basis. It's enough to check that S is linearly independent. Use Wronksian method; since

   $$W[p_1, p_2, p_3, p_4](0) = -60,$$

   we prove the independency. So S is a basis.

2. Vectors in $\mathbb{C}^n$ are of the form

   $$(a_1 + ib_1, a_2 + ib_2, \ldots, a_n + ib_n).$$

   (a) If $\mathbb{C}^n$ has only $\mathbb{R}$ scalars, we can decompose such a vector as

   $$a_1(1, 0, \ldots, 0) + b_1(i, 0, \ldots, 0) + a_2(0, 1, \ldots, 0) + b_2(0, i, \ldots, 0) + \ldots + a_n(0, \ldots, 1) + b_n(0, \ldots, i).$$

   These $2n$ vectors clearly give a basis. So $\mathbb{C}^n$ as a real-vector space has dimension $2n$.

   (b) If $\mathbb{C}^n$ has $\mathbb{C}$ scalars, we can decompose such a vector as

   $$(a_1 + ib_1)(1, 0, \ldots, 0) + (a_2 + ib_2)(0, 1, \ldots, 0) + \ldots + (a_n + ib_n)(0, \ldots, 1).$$

   These n vectors clearly give a basis. So $\mathbb{C}^n$ as a complex-vector space has dimension n.

3. First, we need to compute the following coordinate vectors:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = 0\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + 0\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} + 0\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

Now, the change-of-basis matrix is

$$P_{C\leftrightarrow B} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 1 & 0 & -1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}.$$

4. Reduce $A = \begin{bmatrix}1 & -1 & 2 & 3\\ 1 & 1 & -2 & 6\\ 3 & 1 & 4 & 2\end{bmatrix}$ and get $\begin{bmatrix}1 & -1 & 2 & 3\\ 0 & 1 & -2 & 3/2\\ 0 & 0 & 1 & -13/6\end{bmatrix}$. Thus,

$$\{(1,-1,2,3),(0,1,-2,3/2),(0,0,1,-13/6)\}$$

is a basis for $rowspace(A)$ and

$$\{(1,1,3),(-1,1,1),(2,-2,4)\}$$

is a basis for $colspace(A)$.

5. By rank-nullity theorem, we get $rank(A) + nullity(A) = 7$.
So $rank(A) = 7 - nullity(A)$. By assumption, we have

$$0 \le rank(A) \le 5, \text{ i.e.}$$

$$0 \le 7 - nullity(A) \le 5.$$

This means that

$$2 \le nullity(A) \le 7.$$

For an example of A with $nullity(A) = 2$, i.e. $rank(A) = 5$, we can take

$$A = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}.$$

For an example of A with $nullity(A) = 7$, i.e. $rank(A) = 0$, we get $A = 0$ (the zero matrix).