Homework 8 and Study Problems - MATH 225

In this document, you will find two types of problems: homework and study problems. You are required to submit **only the homework problems** to Gradescope. The study problems are intended to help you grasp the topics thoroughly and prepare for exams. It is strongly advised to attempt all study problems for a comprehensive understanding.

Please submit your homework to Gradescope until **March 29, 11pm**.

Homework problems

1. Determine whether the given function T is a linear transformation or not.
   - $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, 2x - y)$.
   - $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = AB - BA$, where B is fixed $n \times n$ matrix.
   - $T : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(ax^2 + bx + c) = a + b + c + 5$.
   - $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(a, b, c, d) = (a^2, 2b, c^3 + 3d)$.

2. Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations, and let c be scalar. We define **sum** $T_1 + T_2$ and the **scalar product** cT_1 by

$$(T_1 + T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$$

   and

$$(cT_1)(\mathbf{v}) = c(T_1(\mathbf{v}))$$

   for all $\mathbf{v} \in V$. Verify that $T_1 + T_2$ and cT_1 are linear transformations.

3. Consider the linear transformation $S : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $S(A) = A + A^T$ where A is a fixed $n \times n$ matrix. Find $Ker(S)$, $Ran(S)$, and their dimensions.

4. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the vector space V. and suppose that $T_1 : V \to V$ and $T_2 : V \to V$ are the linear transformations satisfying

$$T_1(\mathbf{v}_1) = \mathbf{v}_1 - \mathbf{v}_2, \qquad T_1(\mathbf{v}_2) = 2\mathbf{v}_1 + \mathbf{v}_2,$$
$$T_2(\mathbf{v}_1) = \mathbf{v}_1 + 2\mathbf{v}_2, \qquad T_2(\mathbf{v}_2) = 3\mathbf{v}_1 - \mathbf{v}_2.$$

   Determine $(T_2T_1)(\mathbf{v})$ for an arbitrary vector $\mathbf{v} \in V$.

5. (a) Determine an isomorphism between $\mathbb{R}^2$ and $P_1(\mathbb{R})$.

   (b) Let V denote the vector space of all 4×4 upper triangular matrices. Find n such that V is isomorphic to $\mathbb{R}^n$, and construct an isomorphism.

   (c) Determine an isomorphism between $\mathbb{R}^3$ and the subspace of $M_2(\mathbb{R})$ consisting of all symmetric matrices.

Study problems

1. True-False Review in Pages 388, 389, 405, 416, 417 from the textbook
2. Exercises 6.1.1-23, 6.1.27-36 are good for studying on linear transformations.
3. Exercises 6.3.1-24 are good for studying on the kernel and the range.
4. Exercises 6.3.1-47 are good for studying on the one-to-one, onto, and invertible linear transformations.
5. Additional problems in Pages 429 and 430 (except for 20-23) are good for general study on chapter 6.