Homework 8 Solutions

1. (a) We have

$$\begin{aligned} T(a(x_1,y_1)+b(x_2,y_2)) &= T((ax_1+bx_2, ay_1+by_2)) \\ &= ((ax_1+bx_2)+2(ay_1+by_2), 2(ax_1+bx_2)-(ay_1+by_2)) \\ &= (ax_1+2ay_1+bx_2+2by_2, 2ax_1-ay_1+2bx_2-by_2) \\ &= a(x_1+2y_1, 2x_1-y_1)+b(x_2+2y_2, 2x_2-y_2) \\ &= aT(x_1,y_1)+bT(x_2,y_2). \end{aligned}$$

So T is a linear transformation.

(b) We have

$$\begin{aligned} T(cA_1+dA_2) &= T((cA_1+cA_2)B-B(cA_1+dA_2)) \\ &= cA_1B+dA_2B-BcA_1-BdA_2 \\ &= c(A_1B)-c(BA_1)+d(A_2B)-d(BA_2) \\ &= c(T(A_1))+d(T(A_2)). \end{aligned}$$

So T is a linear transformation.

(c) Since $T(0)=5\neq 0$, T is not a linear transformation.

(d) Since $T(2,0,0)=(4,0,0)$ but $2T(1,0,0)\neq(2,0,0)$, T does not preserve scalar products. So T is not a linear transformation.

2.

$$\begin{aligned} (T_1+T_2)(av_1+bv_2) &= T_1(av_1+bv_2)+T_2(av_1+bv_2) \\ &= aT_1(v_1)+bT_1(v_2)+aT_2(v_1)+bT_2(v_2) \\ &= a(T_1(v_1)+T_2(v_1))+b(T_1(v_2)+T_2(v_2)) \\ &= a(T_1+T_2)(v_1)+b(T_1+T_2)(v_2), \end{aligned}$$

So T_1+T_2 is a linear transformation.

$$\begin{aligned} (cT_1)(av_1+bv_2) &= c(T_1(av_1+bv_2)) \\ &= c(aT_1(v_1)+bT_1(v_2)) \\ &= caT_1(v_1)+cbT_1(v_2) \\ &= a(cT_1)(v_1)+b(cT_1)(v_2), \end{aligned}$$

So cT_1 is a linear transformation.

3.

$$\begin{aligned} \text{Ker}(S) &= \{A\in M_n(\mathbb{R}) \mid S(A)=0\} \\ &= \{A\in M_n(\mathbb{R}) \mid A+A^T=0\} \\ &= \{A\in M_n(\mathbb{R}) \mid A^T=-A\} \\ &= \text{the subspace of skew-symmetric } n\times n \text{ matrices.} \end{aligned}$$

Then

$$\dim(\text{Ker}(S)) = \frac{n^2-n}{2}.$$

By the General Rank-Nullity Theorem, we directly say

$$\dim(\text{Ran}(S)) = n^2 - \left(\frac{n^2-n}{2}\right) = \frac{n^2+n}{2}.$$

On the other hand,

$$\begin{aligned}\text{Ran}(S) &= \{S(A) \mid A \in M_n(\mathbb{R})\} \\ &= \{(A + A^T) \mid A \in M_n(\mathbb{R})\} \\ &= \text{the set of symmetric } n \times n \text{ matrices.}\end{aligned}$$

The last equality holds because

$$B = A + A^T \text{ for some } A \text{ if and only if } B \text{ is symmetric.}$$

4. Let $v \in V$ and write $v = av_1 + bv_2$. Then

$$\begin{aligned}T_2T_1(v) &= T_2T_1(av_1 + bv_2) \\ &= a(T_2T_1)(v_1) + b(T_2T_1)(v_2).\end{aligned}$$

So let's calculate these:

$$\begin{aligned}T_2T_1(v_1) &= T_2(T_1(v_1)) \\ &= T_2(v_1 - v_2) \\ &= T_2(v_1) - T_2(v_2) \\ &= v_1 + 2v_2 - 3v_1 + v_2 \\ &= -2v_1 + 3v_2\end{aligned}$$

$$\begin{aligned}T_2T_1(v_2) &= T_2(T_1(v_2)) \\ &= T_2(2v_1 + v_2) \\ &= 2T_2(v_1) + T_2(v_2) \\ &= 2(v_1 + 2v_2) + (3v_1 - v_2) \\ &= 5v_1 + 3v_2\end{aligned}$$

Therefore,

$$T_2T_1(v) = a(-2v_1 + 3v_2) + b(5v_1 + 3v_2),$$

5. (a) $S : \mathbb{R}^2 \to P_1(\mathbb{R})$
$S((a, b)) = ax + b$
Clearly, it is an isomorphism, namely, an invertible linear transformation.

(b) Any U in V is like

$$U = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & j \\ 0 & 0 & 0 & k \end{bmatrix}$$

So it is easy to compute that $\dim(V) = 10$. This means V is isomorphic to $\mathbb{R}^{10}$. Define $T : V \to \mathbb{R}^{10}$ by

$$T\left(\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & j \\ 0 & 0 & 0 & k \end{bmatrix}\right) = (a, b, c, d, e, f, g, h, j, k).$$

Clearly, this is an isomorphism.

(c) Define $F : \mathbb{R}^3 \to \text{Sym}_n(\mathbb{R})$ as $F(a, b, c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. This is the isomorphism we wanted.