Homework 9 Solutions

1. Given that v is an eigenvector of A with eigenvalue λ, and v is also an eigenvector of B with eigenvalue μ, we want to show:

   (a) v is an eigenvector of the matrix AB. Since

   $$Av = \lambda v$$

   and

   $$Bv = \mu v,$$

   then

   $$ABv = A(Bv) = A(\mu v) = \mu(Av) = \mu(\lambda v) = \lambda\mu v.$$

   Therefore, v is an eigenvector of AB with the corresponding eigenvalue $\lambda\mu$ (equivalently the eigenvalue $\mu\lambda$).

   (b) To prove v is an eigenvector of $A + B$ and find the corresponding eigenvalue, we calculate:

   $$(A + B)v = Av + Bv = \lambda v + \mu v = (\lambda + \mu)v.$$

   Hence, v is an eigenvector of $A + B$, and the corresponding eigenvalue is $\lambda + \mu$.

2. (a) The matrix $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ has two distinct eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 4$, each with multiplicity 1. Since there are two distinct eigenvalues, the matrix A is non-defective in the real field, indicating that it is diagonalizable over $\mathbb{R}$.

   (b) The matrix $A = \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix}$ has a single eigenvalue, $\lambda = 1$, with multiplicity 2. The eigenspace corresponding to this eigenvalue is generated by the eigenvector:

   $$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   Since there is only one linearly independent eigenvector for an eigenvalue of multiplicity 2, the matrix does not have a complete set of linearly independent eigenvectors, indicating that A is defective and not diagonalizable.

   (c) The matrix $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$ has two distinct complex eigenvalues, $\lambda_1 = 2 - 3i$ and $\lambda_2 = 2 + 3i$, each with multiplicity 1. Since there are two distinct eigenvalues, the matrix A is non-defective in the complex field, indicating that it is diagonalizable over $\mathbb{C}$.

3. (a) Expanding the determinant $det(A - \lambda I_3)$ and comparing with the general form of the characteristic polynomial, using the cofactor expansion via the first column, we see that:

$$\begin{aligned} det(A - \lambda I_3) &= (a_{11} - \lambda)((a_{22} - \lambda)(a_{33} - \lambda) - a_{23}a_{32}) + \\ &\quad -a_{21}(a_{12}(a_{33} - \lambda) - a_{11}a_{33}) + \\ &\quad a_{31}(a_{12}a_{23} - a_{13}(a_{22} - \lambda)) \\ &= \ldots + (a_{11} + a_{22} + a_{33})\lambda^2 + \ldots + \\ &\quad (a_{11})(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{11}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= \ldots + (a_{11} + a_{22} + a_{33})\lambda^2 + \ldots + det(A). \end{aligned}$$

Therefore, $b_1 = (a_{11} + a_{22} + a_{33})$, and $b_3 = \det(A)$. Here, b_1 is the coefficient of λ^2 and b_3 is the constant term of the polynomial, which equals the determinant of A.

(b) By expanding the polynomial in terms of eigenvalues, we find:

$$p(\lambda) = \lambda_1\lambda_2\lambda_3 - (\lambda_1 + \lambda_2)\lambda_3\lambda + \lambda_3\lambda^2 - (\lambda_1\lambda_2)\lambda + (\lambda_1 + \lambda_2)\lambda^2 - \lambda^3,$$

and so

$$\begin{aligned} b_1 &= (\lambda_1 + \lambda_2 + \lambda_3), \\ b_3 &= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

This shows that b_1 is the sum of the eigenvalues, and b_3 is the product of the eigenvalues.

(c) From parts (a) and (b), we can conclude:

$$\begin{aligned} \det(A) &= b_3 = \lambda_1\lambda_2\lambda_3, \\ \text{tr}(A) &= b_1 = a_{11} + a_{22} + a_{33} = \lambda_1 + \lambda_2 + \lambda_3. \end{aligned}$$

Therefore, the determinant of A is the product of its eigenvalues, and the trace of A is the sum of its eigenvalues.

4. (a) Let $P = I$, where I is the identity matrix. Since I is invertible and $I^{-1} = I$, we have:

$$I^{-1}AI = A.$$

Thus, A is similar to itself by definition, proving reflexivity.

(b) Given that A is similar to B, there exists an invertible matrix P such that:

$$P^{-1}AP = B.$$

To show B is similar to A, consider P^{-1} as the invertible matrix. Multiplying both sides of the equation by $P = (P^{-1})^{-1}$ from the left and P^{-1} from the right, we get:

$$(P^{-1})^{-1}BP^{-1} = PBP^{-1} = A.$$

Hence, B is similar to A with the invertible matrix P^{-1}, proving symmetry.

(c) Given that A is similar to B, there exists an invertible matrix P such that:

$$P^{-1}AP = B.$$

And if B is similar to C, there exists an invertible matrix Q such that:

$$Q^{-1}BQ = C.$$

Substituting the expression for B gives:

$$(PQ)^{-1}A(PQ) = Q^{-1}P^{-1}APQ = Q^{-1}BQ = C.$$

Since the product of invertible matrices is invertible, and $(PQ)^{-1} = Q^{-1}P^{-1}$, we have shown that A is similar to C using the invertible matrix PQ, proving transitivity.

5. (a) Given the matrix A:

$$A = \begin{bmatrix} 1 & -3 & 3 \\ -2 & -4 & 6 \\ -2 & -6 & 8 \end{bmatrix},$$

we found its eigenvalues to be $\lambda_1 = 1$ with multiplicity 1, and $\lambda_2 = 2$ with multiplicity 2. Details are left to the student, but the corresponding eigenvectors are:

- For $\lambda_1 = 1$, an eigenvector is $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$.
- For $\lambda_2 = 2$, two linearly independent eigenvectors are $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Using these eigenvectors, we construct the matrix S as:

$$S = \begin{bmatrix} \frac{1}{2} & -3 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then, we calculate $S^{-1}AS$ and verify it yields a diagonal matrix:

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where the diagonal elements are the eigenvalues of A, confirming that S diagonalizes A.

(b) Given the matrix A:

$$A = \begin{bmatrix} 3 & -2 & 3 & -2 \\ -2 & 3 & -2 & 3 \\ 3 & -2 & 3 & -2 \\ -2 & 3 & -2 & 3 \end{bmatrix},$$

we found its eigenvalues to be $\lambda_1 = 10$, $\lambda_2 = 2$, and $\lambda_3 = 0$ (with multiplicity 2). Details are left to the student, but the corresponding eigenvectors are:

- For $\lambda_1 = 10$, an eigenvector is $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.
- For $\lambda_2 = 2$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.
- For $\lambda_3 = 0$, two linearly independent eigenvectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Using these eigenvectors, we construct the matrix S as:

$$S = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then, we calculate $S^{-1}AS$ and verify it yields a diagonal matrix:

$$S^{-1}AS = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the diagonal elements are the eigenvalues of A, confirming that S diagonalizes A.