1. (20 points) Examine the propositions below and determine whether they are true or false. You do not need to provide an explanation, just mark your choice. (Each is 4 points)

True — False ✓ The set $\{0\}$, with the usual operations of addition and scalar multiplication, forms a vector space.

True — False ✓ The set of column vectors of a 5×7 matrix A must be linearly dependent.

True — False ✓ A change-of-basis matrix is always a square matrix.

True — False ✓ If A is a 7×9 matrix with $nullity(A) = 2$, then $rowspace(A) = \mathbb{R}^7$.

True — False ✓ If $T : \mathbb{R}^8 \to \mathbb{R}^3$ is an onto linear transformation, then $Ker(T)$ is five-dimensional.

dim Ran(T) = W dim(Ran(T)) > 3

2. Prove the following statements

(a) (10 points) Let $T : V \to W$ be a linear transformation, and assume that V and W are both finite dimensional. If T is one-to-one, then

$$dim(V) \leq dim(W).$$

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Since T is one to one $Ker(T) = \{0\}$ $dim(Ker(T)) = 0$.

According to general rank nullity theorem

$dim(Ran(T)) + dim(Ker(T)) = dim(V)$

$\therefore dim(Ran(T)) = dim(V)$

$\because Ran(T) \subseteq W$

$\therefore dim(Ran(T)) \leq dim(W)$

$\therefore dim(V) \leq dim(W)$ ✓

(b) (10 points) Let $T : V \to W$ be a linear transformation. Then

$$Ran(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$$

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is a subspace of W.

Let $\vec{w}, \vec{z} \in Ran(T)$ $c \in \mathbb{R}$, there're $\vec{u}, \vec{v} \in V$ so that $T(\vec{u}) = \vec{w}$ $T(\vec{v}) = \vec{z}$

Addition: $\vec{w} + \vec{z} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}) \in Ran(T)$

Scalar multipilcation: $c\vec{w} = cT(\vec{u}) = T(c\vec{u}) \in Ran(T)$ ✓

$\therefore$ Close under both addition & scalar multiplication

$\therefore Ran(T) \subseteq W$

3. Consider the following set of polynomials

$$\begin{aligned} S_1 &= \{2x + 5x^2, 1 + x\} \\ S_2 &= \{1 - x + x^2, -1 + 2x - x^2, 2x + x^2\} \\ S_3 &= \{2, 4x, 5x^2, 8x + x^2\} \end{aligned}$$

(a) (5 points) Which one can be a basis for $P_2(\mathbb{R})$?

$\dim(P_2(\mathbb{R})) = 3$

number of vectors in $S_1 < 3$ $\therefore$ S_1 can't be spanning set

number of vectors in $S_3 > 3$ $\therefore$ S_3 is linearly dependent

number of vectors in S_2 = $\dim(P_2(\mathbb{R})) = 3$

$\therefore$ S_2 can be a basis for $P_2(\mathbb{R})$

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(b) (10 points) For your guess in part (a), use Wronskian method to show the set is linearly independent.

$$W[f_1\ f_2\ f_3](x) = \det\left(\begin{bmatrix} 1-x+x^2 & -1+2x-x^2 & 2x+x^2 \\ -1+2x & 2-2x & 2+2x \\ 2 & -2 & 2 \end{bmatrix}\right)$$

when $x=0$ $W[f_1\ f_2\ f_3](0) = \det\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 2 & -2 & 2 \end{bmatrix}\right) = 4-4+4-2 = 2 \neq 0$

$\therefore$ S_2 is linearly independent

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(c) (5 points) Why does part (b) finish to prove that your guess is indeed a basis? Explain.

Because for a set of vectors $\{v_1 \ldots v_k\}$ if $k = \dim(V)$

it's enought to prove this set of vectors is linearly independent or

spanning set of V to prove it's a basis of V

In this case, number of vectors in $S_2 = 3 = \dim(P_2(\mathbb{R}))$ & S_2 is linearly independent

$\therefore$ S_2 is a basis of $P_2(\mathbb{R})$

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4. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(a, b, c) = (a - b, 3c).$$

(a) (7 points) Find the matrix representation of T.

$\mathbb{R}^3 = \text{span}\{(1,0,0)(0,1,0)(0,0,1)\}$

$T(1,0,0) = (1,0)$

$T(0,1,0) = (-1,0)$

$T(0,0,1) = (0,3)$

$A_{2\times 3} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

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(b) (8 points) Find $Ker(T)$ and $Ran(T)$.

$Ker(T) = \text{nullspace}(A)$

for $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow{M_2(\frac{1}{3})} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\begin{cases} x - y = 0 \\ z = 0 \end{cases}$ $\begin{cases} x = y \\ z = 0 \end{cases}$

$Ker(T) = \text{nullspace}(A) = \text{span}\{(1,1,0)\}$

$Ran(T) = \text{columnspace}(A) = \text{span}\{(1,0)(0,3)\}$

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(c) (5 points) Determine whether T is one-to-one and/or onto?

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$\because Ker(T) = \text{span}\{(1,1,0)\} \neq \{(0,0,0)\}$ T isn't one to one

$\because \dim(Ran(T)) = 2$ $Ran(T) \subseteq \mathbb{R}^2$

$\therefore Ran(T) = \mathbb{R}^2$

T is onto

5. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 3 & 1 & 8 \end{bmatrix}$.

(a) (4 points) Define *rowspace*(A) and *colspace*(A). definition

rowspace(A) is span$\{r_1 \dots r_m\} \subseteq \mathbb{R}^n$ for $A_{m \times n}$

Colspace(A) is span$\{c_1 \dots c_n\} \subseteq \mathbb{R}^m$ for $A_{m \times n}$

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(b) (6 points) Reduce A into the reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 3 & 1 & 8 \end{bmatrix} \xrightarrow[A_{13}(-3)]{A_{12}(-1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cancel{\xrightarrow[A_{23}(-1)]{A_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{M_3(\frac{1}{2})} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\xrightarrow{A_{23}(-1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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(c) (10 points) Find the bases for *rowspace*(A) and *colspace*(A).

rowspace(A) = span$\{(1,0,2)\,(0,1,2)\}$ basis is $\{(1,0,2)\,(0,1,2)\}$

Colspace (A) = span$\{(1,1,3)\,(0,1,1)\}$ basis is $\{(1,1,3)\,(0,1,1)\}$

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