Functions

Definition A function f is a rule that maps each element in a set D to exactly one element, called $f(x)$, in a set E.

! By a set, we mean a collection of some entities. Some famous ones are

$\mathbb{N}$ - natural numbers

$\mathbb{Z}$ - integers

$\mathbb{Q}$ - rational numbers

$\mathbb{R}$ - real numbers

"$x \in D$" means "x is an element of the set D".

! We can denote the function as

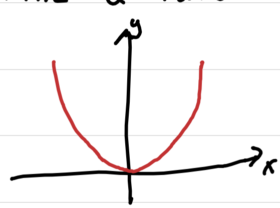
$$f : D \to E$$
$$x \mapsto f(x)$$

! Here, "exactly one" is the key part. In a function, an element in D cannot go to two distinct elements.

! We call D "domain", E "range", x "input", and $f(x)$ "output".

Remark We can either write an explicit formula or sketch its graph when we define a function.

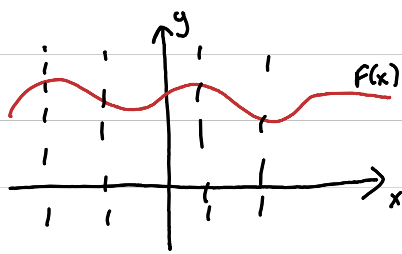
Example $f(x) = x^2$ has graph



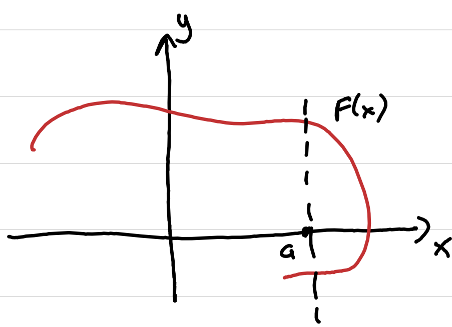
• Note that we can use different variables, namely, $f(t)$, $g(x)$, $H(u)$, etc. we can change the letters for functions or inputs.

For $f(x)$, we say "f is a function of x".

<u>Remark</u> We can test whether given mapping is a function or not via its graph



This is a function since each x has exactly one output



This is NOT a function since a has two outputs.

<u>Domain of a function</u> In some cases, a function may be undefined at some points. For example, $f(x) = \frac{1}{x}$ is not defined at $x = 0$. We write its domain as

$$\mathbb{R} \setminus \{0\} \quad \text{or} \quad (-\infty, 0) \cup (0, \infty).$$

Example $f(x) = \dfrac{1}{x^2-9}$.

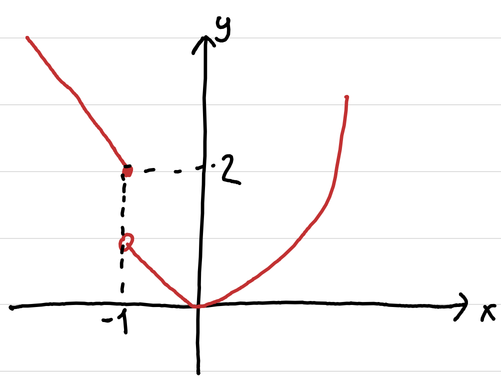
Recall that $x^2-9=(x-3)(x+3)$, so we have
$$f(x) = \frac{1}{(x-3)(x+3)}.$$

The denominator cannot be zero, so $x-3 \neq 0$ or $x+3 \neq 0$. The domain can be written as
$$\mathbb{R} \setminus \{-3,+3\} \quad \text{or} \quad (-\infty,-3) \cup (-3,3) \cup (3,\infty).$$

Piecewise defined functions

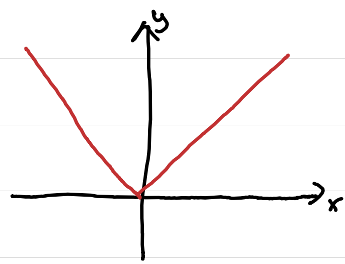
Example

$$f(x) = \begin{cases} 1-x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$



Example: Absolute value function is an example of piecewise definition:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Even or odd functions

Definitions A function $f: D \to E$ is called even if

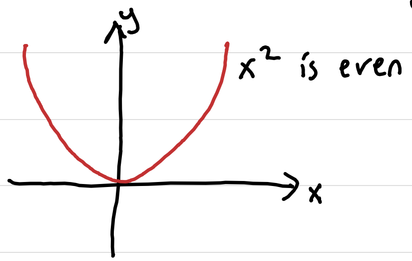
$$f(-x) = f(x) \quad \text{for all } x \in D,$$

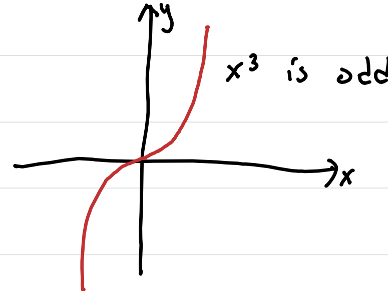
it is called odd if

$$f(-x) = -f(x) \quad \text{for all } x \in D.$$

! A function can be neither of them.

! Even functions are symmetric with respect to y-axis, and odd functions are symmetric about the origin.





Example Determine if $f(x) = x^3 + 5x$ is even or odd.

Solution Let x be arbitrary.

$$\begin{aligned} f(-x) &= (-x)^3 + 5(-x) \\ &= -x^3 - 5x \\ &= -(x^3 + 5x) = -f(x). \end{aligned}$$

So f is an odd function.

Increasing and Decreasing Functions

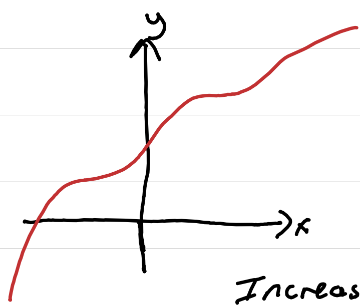
Definition A function $f : I \to \mathcal{U}$ is called

<u>increasing on I</u> if

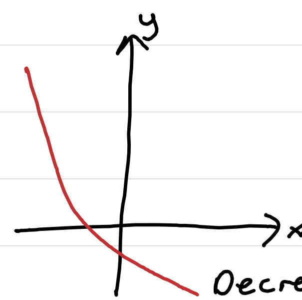
$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I,$$

<u>decreasing on I</u> if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I.$$



Increasing



Decreasing

Special Functions

(1) Polynomials

A function $P(x)$ is called polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are numbers. These are called coefficients. The highest power n is called the degree.

If $n=1$, $P(x)$ is called linear function,
$n=2$, $P(x)$ is called quadratic function,
$n=3$, $P(x)$ is called qubic function.

! The domain of any polynomial is $\mathbb{R}$, that is, polynomials are defined everywhere.

Example $3x+5$, x^2+4, x^3+7x+8, $x^{17}+x^3+5x$, etc.

(2) Power Functions

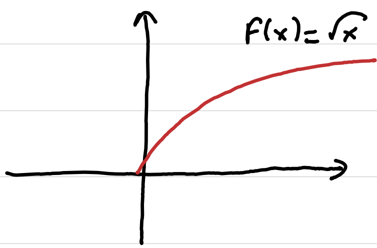
$f(x) = x^a$ where a is constant.

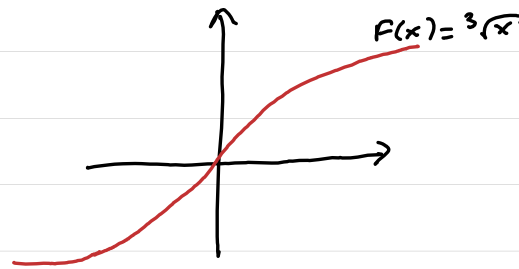
Case 1 $a=n$ where n is a positive integer.

Then the domain is $\mathbb{R}$. If n is an even/odd number, then x^a is an even/odd function.

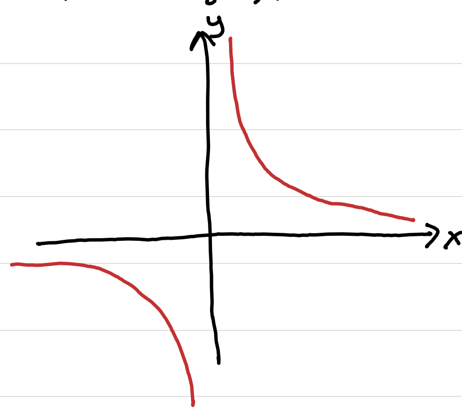
Case 2 $\quad a = \frac{1}{n}$, where n is a positive integer.

Then $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is called root function.
For even number n, the domain is $[0, \infty)$.
For odd number n, the domain is $\mathbb{R}$.





Case 3 $\quad a = -1$, namely, $f(x) = \frac{1}{x}$. This is defined on $\mathbb{R} \setminus \{0\}$.



③ Rational Functions $\quad f(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials.

Whose domain is the set of elements x such that $Q(x) \neq 0$.

Example $f(x) = \dfrac{3x+5}{7x-4}$ is a rational function.

It is defined if $7x-4 \neq 0$, so $x \neq \frac{4}{7}$. The domain is $\mathbb{R} \setminus \{\frac{4}{7}\}$.

④ **Algebraic Functions**

An algebraic function f is obtained via algebraic operations on polynomials like $+, -, *, \div$, and taking roots. The domain can be found via analysis on its part.

Example $f(x) = \sqrt{x+1} + \dfrac{1}{x-5}$

- For $\sqrt{x+1}$, we need $x+1 \geq 0$, so $x \geq -1$.
- For $\dfrac{1}{x-5}$, we need $x-5 \neq 0$, so $x \neq 5$.

- This means that the domain is $[-1, \infty) \setminus \{5\}$.

⑤ **Exponential Functions** $f(x) = b^x$ where b is a positive real number.

Rules

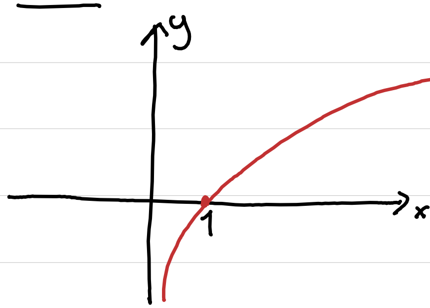
- $b^{x+y} = b^x b^y$
- $b^{x-y} = \dfrac{b^x}{b^y}$
- $(b^x)^y = b^{xy}$
- $(ab)^x = a^x b^x$

Remark
- If $0<b<1$, then f is decreasing.
- If $b=1$, then $f(x)=1$ everywhere.
- If $b>1$, then f is increasing.

There is a special number "e" like "π" (pi) which has value ≈ 2.71828. The function $f(x)=e^x$ is called natural exponential, and it has a special importance (later).

⑥ Logarithmic Functions

$f(x)=\log_b x$ where b is a positive number and called the base. To be defined, we must have $x>0$.



Rules:
- $\log_b 1 = 0$
- $\log_b (xy) = \log_b x + \log_b y$
- $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
- $\log_b (x^r) = r \log_b x$
- $\log_b x = \dfrac{\ln x}{\ln b}$ (Change of base Formula)

When $n=e$, we write $\log_e x = \ln x$ and call it natural logarithm.

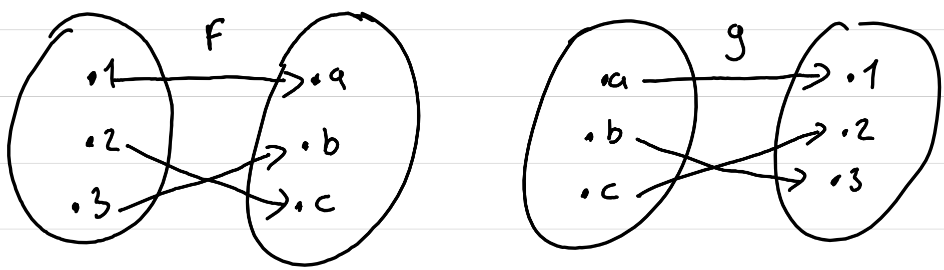
<u>Example</u> $\log_3(x-5)$ is defined for $x-5>0$. So the domain is $(5,\infty)$.

<u>Inverse Functions</u>

If $f: D \to E$ is a function, under some conditions, we can reverse the mapping; namely, there is a $g: E \to D$ such that

$$f(x) = y \iff g(y) = x.$$

<u>Example</u>



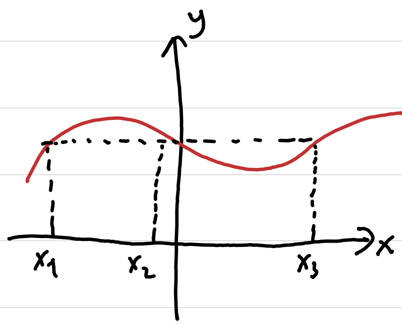
f and g are inverses of each other.

We need the following condition.

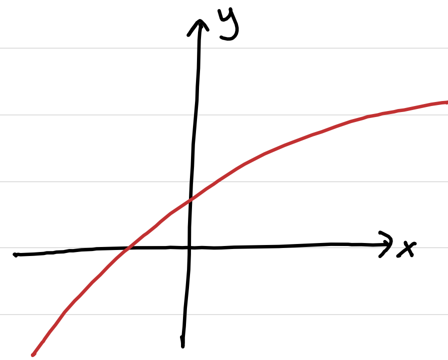
<u>Definition</u> A function $f: D \to E$ is called <u>one-to-one</u> if it never takes on the same value twice;

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

Remark To have an inverse function, we also need that f must be onto (no missing output in range), but we assume all functions are onto in this course.



not one-to-one since x_1, x_2, x_3 have same output



one-to-one since each output appears once

Example • $f: \mathbb{R} \to [0, \infty)$
$x \mapsto x^2$ is not one-to-one since $f(-1) = 1 = f(1)$.

• $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto 3x+5$ is one-to-one. Let's show:

We want to show $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. This is equivalent to say if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Suppose $f(x_1) = f(x_2)$, namely, we have

$$3x_1 + \cancel{5} = 3x_2 + \cancel{5}$$
$$\cancel{3}x_1 = \cancel{3}x_2$$
$$x_1 = x_2.$$

So we are done.

<u>Definition</u> Let $f: A \longrightarrow B$ be one-to-one. Then its <u>inverse</u>, denoted by f^{-1}, is a function from B to A defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any $y \in B$.

! Be careful. f^{-1} and $\frac{1}{f}$ are different things.

<u>Remarks</u>
- $f^{-1}(f(x)) = x$ for every $x \in A$.
- $f(f^{-1}(y)) = y$ for every $y \in B$.

<u>Example</u>
- $f(x) = x^3$ is one-to-one since if $x_1^3 = x_2^3$ then $x_1 = x_2$. So it has an inverse and given by $f^{-1}(x) = x^{\frac{1}{3}}$.
- b^x and $\log_b x$ are inverses of each other. In other words, if $f(x) = b^x$, then $f^{-1}(x) = \log_b x$, and vice versa.

How to find the inverse for given f? Let's explain via an example:

Let $f: [0,\infty) \to [5,\infty)$

$$x \mapsto x^2+5$$

Step ① Write $y = f(x)$

In our case, we have $y = x^2+5$.

Step ② Solve x in terms of y

In our case, we have

$$y-5 = x^2$$
$$\sqrt{y-5} = x$$

Step ③ To express f^{-1}, interchange x and y.

In our case, we have

$$f^{-1}(x) = \sqrt{x-5}.$$

Also f^{-1} has domain $[5,\infty)$,

range $[0,\infty)$.

Example Find the inverse of $f: \mathbb{R} \setminus \{-\frac{5}{2}\} \to \mathbb{R}$

$$x \mapsto \frac{1}{2x+5}$$

Step ① Let $y = \dfrac{1}{2x+5}$,

Step 2 $\quad y(2x+5) = 1$

$$2x+5 = \frac{1}{y}$$

$$2x = \frac{1}{y} - 5 = \frac{1-5y}{y}$$

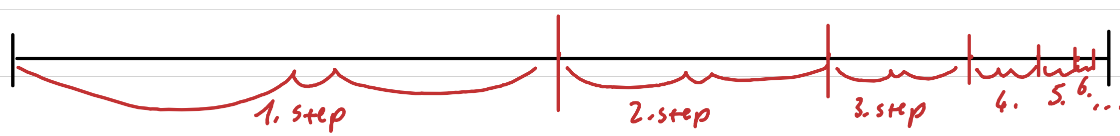
$$x = \frac{1-5y}{2y}$$

Step 3 $\quad F^{-1}(x) = \frac{1-5x}{2x}$.

Limits

Imagine you are standing on one side of a room, and there is a wall on the opposite side. Your goal is to walk towards the wall, but there is a catch: each step you take must be half the distance of the previous step. So, if the room is 16 feet long, your first step would be 8 feet, the next step would be 4 feet, then 2 feet, and so on.

As you continue walking, each steps gets smaller and smaller. Even though you keep moving closer and closer to the wall, you never quite reach it because there is always some distance left to cover, no matter how small. You can get as close to the wall as you like, but you will never actually touch it.



Now, think of the wall as a specific value you are trying to reach, which we'll call 'L'. Your steps represent the values of a function, f(x), and your distance from the wall represents how close these values get to 'L'.

This is similar to what happens with limits. When we say the limit of a function as x approaches a certain point is L, we mean that by making x close enough to that point, the value of the function can be made as close to 'L' as we want, even though it may never actually reach 'L'. Just like you can get infinitely close to the wall but never touch it, the function can get infinitely close to 'L' without necessarily being equal to 'L'.

Definitions

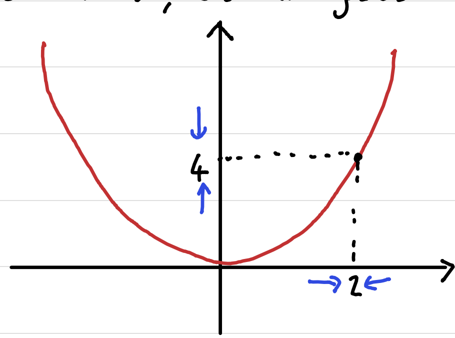
① Suppose $F(x)$ is defined near the value 'a', and as x gets closer to a, the value of $F(x)$ gets closer to L. Then we write

$$\lim_{x \to a} F(x) = L$$

and say

"the limit of $F(x)$, as x goes to a, equals L".

Examples

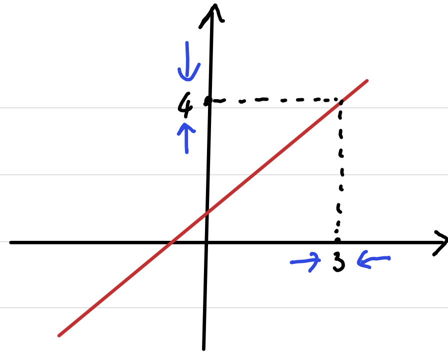


$F(x) = x^2$

When x gets closer to 2, $F(x)$ gets closer to 4.

$$\boxed{\lim_{x \to 2} x^2 = 4.}$$

Examples



$f(x) = x+1$

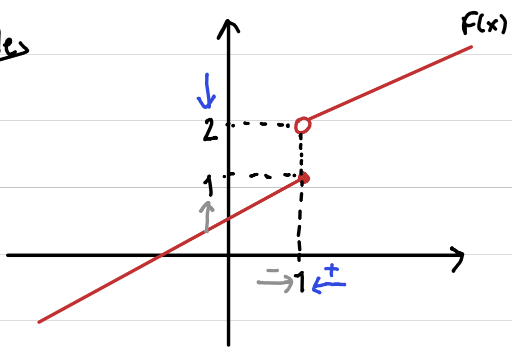
$$\boxed{\lim_{x \to 3} (x+1) = 4}$$

(2) A <u>one-sided</u> limit refers to the value a function approaches as the input 'x' approaches a specific point 'a' from only one side - either from the left or the right.

$\lim_{x \to a^-} f(x) = L$ "the limit of $f(x)$, as x goes to a <u>from the left</u>, is L"

$\lim_{x \to a^+} f(x) = L$ "the limit of $f(x)$, as x goes to a <u>from the right</u>, is L"

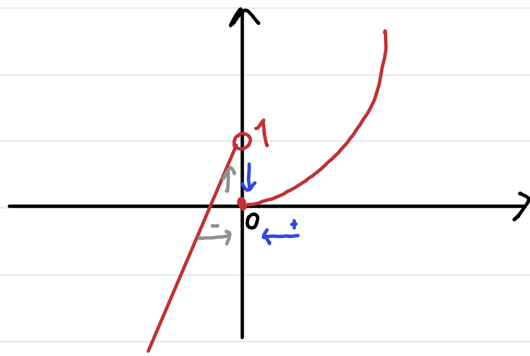
Examples



$\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 2$

Example: $f(x) = \begin{cases} 2x+1 & \text{if } x<0 \\ x^2 & \text{if } x \geq 0 \end{cases}$.



$\lim_{x \to 0^-} f(x) = 1$

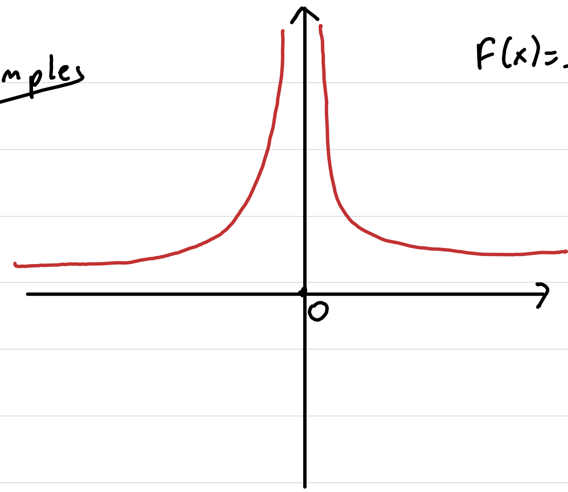
$\lim_{x \to 0^+} f(x) = 0$

③ An <u>infinite limit</u> occurs when the values of $f(x)$ increase or decrease without bound as x gets closer to a specific point a.

$\lim_{x \to a} f(x) = \infty$ means the value of $f(x)$ can be made arbitrarily large by taking x close enough to a.

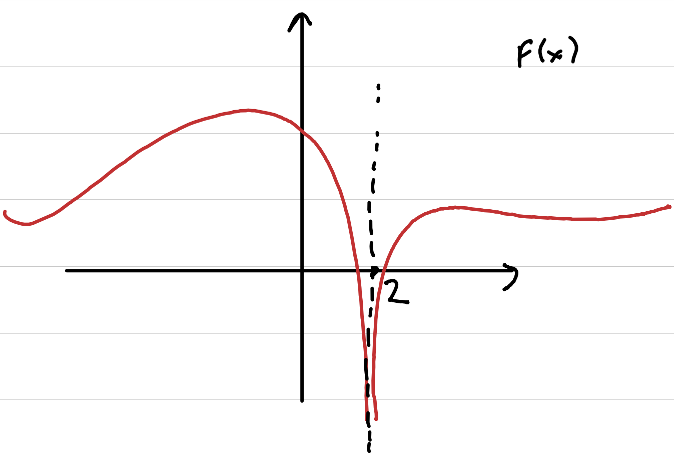
$\lim_{x \to a} f(x) = -\infty$ means the value of $f(x)$ can be made arbitrarily small by taking x close enough to a.

Examples



$F(x) = \frac{1}{x^2}$

$$\boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty}$$

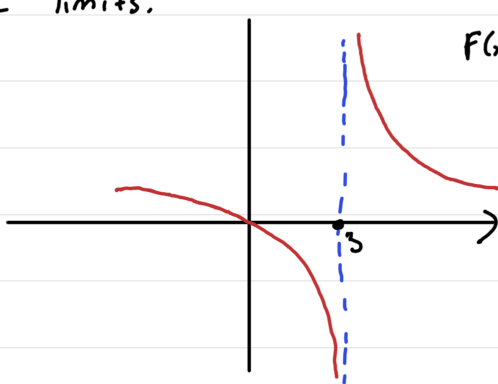


$F(x)$

$$\lim_{x \to 2} F(x) = -\infty$$

Remark. We can also talk about <u>one-sided</u> infinite limits.

Examples



$F(x) = \frac{2x}{x-3}$

$$\boxed{\lim_{x \to 3^+} F(x) = \infty}$$

$$\boxed{\lim_{x \to 3^-} F(x) = -\infty}$$

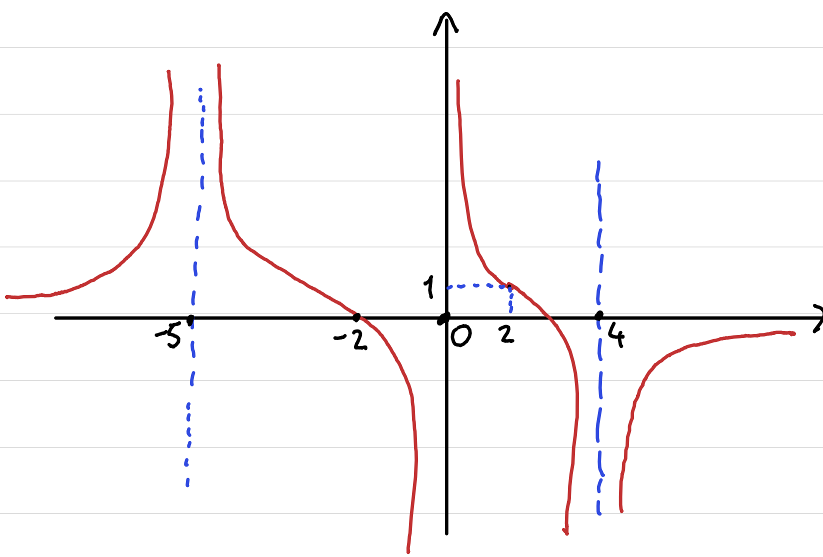
Definition The vertical line $x=a$ is called a **vertical asymptote** of the curve $y=f(x)$ if at least one of the following is true:

$$\lim_{x\to a} f(x)=\infty \qquad \lim_{x\to 0^+} f(x)=\infty \qquad \lim_{x\to a^-} f(x)=\infty$$

$$\lim_{x\to a} f(x)=-\infty \qquad \lim_{x\to a^+} f(x)=-\infty \qquad \lim_{x\to a^-} f(x)=-\infty$$

In class exercise

Find the following limits of the function $g(x)$ given below.



$\lim_{x\to 5} g(x)=?$ $\qquad \lim_{x\to -2} g(x)=?$ $\qquad \lim_{x\to 0^-} g(x)=?$ $\qquad \lim_{x\to 0^+} g(x)=?$

$\lim_{x\to 4} g(x)=?$ $\qquad \lim_{x\to 2^-} g(x)=?$

Limit Laws

Suppose that the limits

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists (but not $\pm\infty$).

1. $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

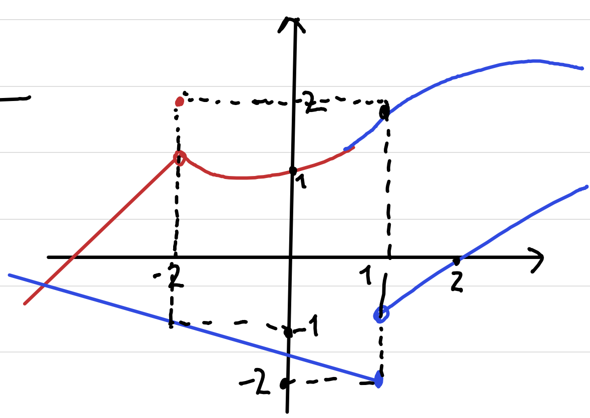
2. $\lim_{x\to a} (f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. For a constant c, $\lim_{x\to a} cf(x) = c\left(\lim_{x\to a} f(x)\right)$

4. $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ when $\lim_{x\to a} g(x) \neq 0$.

Example



We have

- $\lim_{x\to -2} f(x) + 5g(x) = \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x)$

$$= 1 + 5(-1) = -4$$

- $\lim_{x\to 1} f(x)g(x)$ is undefined since $\lim_{x\to 1} g(x)$ does not exist.

- $\lim_{x\to 2} \frac{f(x)}{g(x)}$ is undefined since $\lim_{x\to 2} g(x) = 0$.

6. $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$ for all positive integer n.

7. $\lim_{x\to a} c = c$

8. If $f(x)$ is a polynomial or a rational function and a is in the domain of f, then

$$\boxed{\lim_{x\to a} f(x) = f(a),}$$

Example: $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1}$. This is a rational function but direct substitution gives $\dfrac{0}{0}$, so we first simplify the ratio (if possible).

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x \to 1} x+1 = 2.$$

Example: $\lim\limits_{x \to 0} \dfrac{2x+5}{x^2+3x+7} = \dfrac{2(0)+5}{(0)^2+3(0)+7} = \dfrac{5}{7}$.

Theorem: $\lim\limits_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

Example: Show that $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

We have $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} \dfrac{-x}{x} = \lim\limits_{x \to 0^-} -1 = \underline{\underline{-1}}$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = \underline{\underline{1}}.$$

Since $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} \neq \lim\limits_{x \to 0^+} \dfrac{|x|}{x}$, the limit does not exist.

Theorem If $f(x) \leq g(x)$ when x is near a and $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ exist, we have

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ near a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then $\lim_{x \to a} g(x) = L$.

Recall the trig functions like sin, cos, tan, cot, etc.

! We'll share a note about them, please read.

Examples Find $\lim_{x \to 0} x^2 \left(\sin \frac{1}{x}\right)$.

Recall $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$. Multiplying each side with x^2, we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Since $\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$, by Squeeze Theorem, $\boxed{\lim_{x \to 0} f(x) = 0.}$

Continuity

Recall that if $p(x)$ is a polynomial, then we get $\lim_{x \to a} p(x) = p(a)$. Many essential functions have the same property, and we call such functions "continuous".

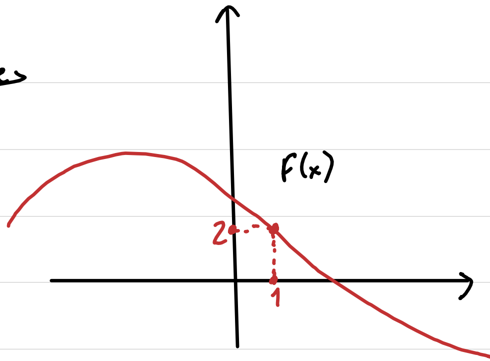
Definition A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

Remarks The definition actually says more than it appears. To say f is continuous at a, we need

- $f(a)$ is defined,
- $\lim_{x \to a} f(x)$ exists, and
- $\lim_{x \to a} f(x) = f(a)$.

So we need to check all these three conditions to determine continuity at a point.

Examples



$f(x)$ is continuous at 1 since $f(1)=2$ and $\lim_{x\to 1} f(x) = 2$,

Examples The function $f(x) = \frac{1}{x}$ is not continuous at 0 since $f(0)$ is not defined.

Examples Define $f(x) = \begin{cases} x^2+2 & x>1 \\ 5 & x=1 \\ 3x & x<1 \end{cases}$,

Then $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+2 = 3$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x = 3$,

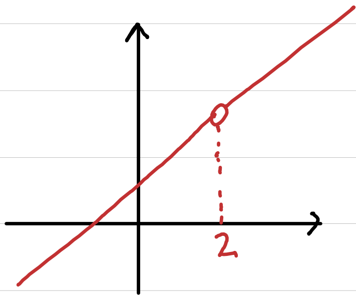
so we have $\lim_{x\to 1} f(x) = 3$, However, $f(1)=5$.

Therefore, f is not continuous at 1.

Remark If f is not continuous at a, we say f is <u>discontinuous</u> at a.

There are four cases for discontinuity. We'll look into these via examples

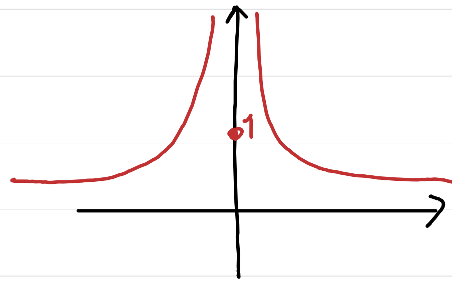
(1) $f(x) = \dfrac{x^2-x-2}{x-2}$



Removable discontinuity

since $f(2)$ is undefined but it can be removed via redefining.

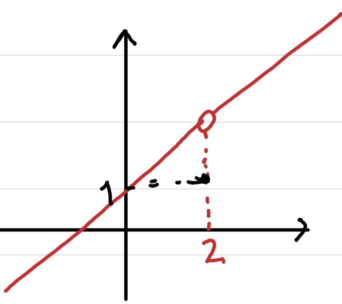
(2) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



Infinite discontinuity

since $f(0) = 1$ but $\lim\limits_{x \to 0} f(x) = \infty$

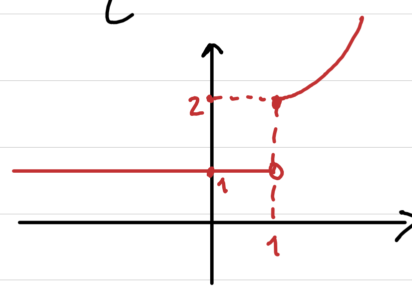
(3) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



Removable discontinuity

since $\lim\limits_{x \to 2} f(x) \neq f(2)$ but it can be removed via redefining.

(4) $f(x) = \begin{cases} x^2+1 & x \geq 1 \\ 1 & x < 1 \end{cases}$



Jump discontinuity

since $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$.

Remarks • If we have $\lim_{x\to a^+} f(x) = f(a)$, we say

"f is continuous from the right at a".

• If we have $\lim_{x\to a^-} f(x) = f(a)$, we say

"f is continuos from the left at a".

• If f is continuous at every point a in an interval I, we say

"f is continuous on the interval I",

Examples $f(x) = \dfrac{x-1}{3x+6}$ is continuous on $(-\infty, -2)$

since if $a < -2$

$$\lim_{x\to a} \frac{x-1}{3x+6} = \frac{\lim_{x\to a} x-1}{\lim_{x\to a} 3x+6} = \frac{a-1}{3a+6} = f(a).$$

Similarly, $f(x)$ is continuous on $(-2,\infty)$. However, for example, $f(x)$ is not continuous on $(-3,3)$ since -2 is in $(-3,3)$ but f is discontinuous at -2.

Remark. Similar to limit laws, we have if f and g are continuous at a, then

$$f+g,\ f-g,\ fg,\ cf,\ \frac{f}{g}\ (g(a)\neq 0)$$

are continous at a.

- All polynomials are continuous on $\mathbb{R}$.
- All rational functions are continuous on their domain.

Actually, we have more than polynomials and rational functions.

Theorem. The following types of functions are continuous at every number in their domain:

- algebraic functions,
- trigonometric functions,
- inverse trigonometric functions,
- exponential functions,
- logarithmic functions.

! The proof is highly detailed and beyond the scope of this course. The rough idea is that any such function is close enough to a polynomial. Since we have continuity for polynomials, we have the same for any function close to a polynomial.

Digression Recall that if there are functions

$g: A \to B$ and $f: B \to C$, then we can combine them and get a new function

$$f \circ g : A \to C$$

defined as $f \circ g(x) = f(g(x))$. It is called the composition of f and g.

The relation between continuous functions and compositions is useful in computations:

Theorem ① If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f \circ g(x) = f\left(\lim_{x \to a} g(x) \right) = f(b),$$

② If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

For example, since we know $\sin(x)$ and x^3+5 are continuous functions, their compositions in both ways

$$\sin(x^3+5) \quad \text{and} \quad (\sin(x))^3+5$$

are also continuous functions.

! Be careful about the domains. For example, $1+\cos(x)$ is continuous everywhere, but $\ln x$ is continuous on $(0,\infty)$. If we consider the composition $\ln(1+\cos(x))$, we need $1+\cos(x)>0$. In other words $\cos(x)>-1$. So whenever $\cos(x)=-1$, the composition is undefined and hence not continuous. If $\cos(x)=-1$, then $x=\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

Therefore,

$$\ln(1+\cos(x)) \text{ is continuous everywhere}$$

except for $x=\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

Example: Find $\lim_{x\to 1} \ln(x^2-x+e)$.

Solution: Since $\ln$ is continuous, we have

$$\lim_{x\to 1} \ln(x^2-x+e) = \ln\left(\lim_{x\to 1} x^2-x+e\right)$$
$$= \ln(1-1+e) = \ln e = 1$$

__Examples__ Let us explain why

$$\frac{\sin(x) + e^{(x^2+3)}}{\sqrt{x-1}}$$

is a continuous function on its domain.

1. $\sin(x)$ is continuous as a trig function.
2. $\sqrt{x-1}$ is continuous as an algebraic function.
3. x^2+3 is continuous as a polynomial.
4. e^x is continuous as an exponential.
5. e^{x^3+3} is continuous as a composition
6. $\sin(x) + e^{x^2+3}$ is continuous as an addition
7. $\dfrac{\sin(x)+e^{(x^2+3)}}{\sqrt{x-1}}$ is continuous as a ratio.

! In other words, we are using algebraic and compositional rules for deciding continuity.

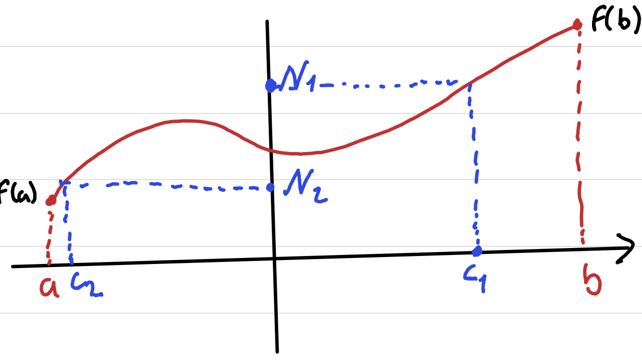
The following theorem is very essential for many applications, especially for solving equations.

Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there is a number c such that

$$a < c < b \quad \text{and} \quad f(c) = N.$$

We can illustrate the idea as follows:



We can pick any N between $f(a)$ and $f(b)$, and since f is continuous, it is like there is a thread between these points. We can follow the thread and find suitable input c such that $f(c) = N$.

Examples Show that $4x^3-6x^2+3x-2=0$ has a root between 1 and 2.

Solution: Let us define $f(x)=4x^3-6x^2+3x-2$.
Then f is a polynomial, so continuous.
Also, $f(1)=4-6+3-2=-1$ and
$f(2)=4\cdot 8-6\cdot 4+3\cdot 2-2=32-24+6-2=12$.
In other words, we have

$$f(1)<0<f(2),$$

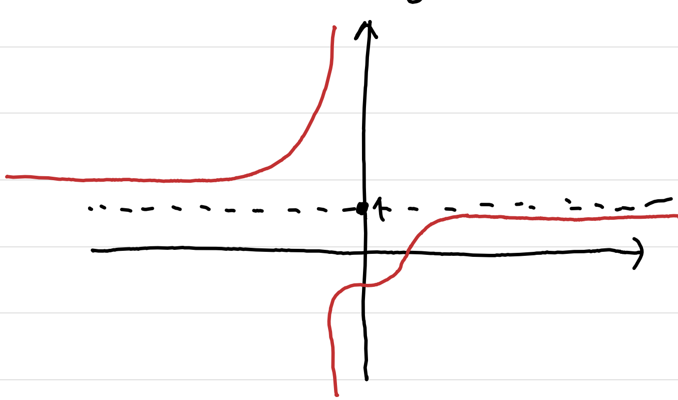
By IMT, there is a number c such that

$$1<c<2 \quad \text{and} \quad f(c)=0.$$

This c is the root we wanted.

Limits at Infinity / Horizontal Asymptotes

Consider the following situation: $f(x)=\dfrac{x^3-1}{x^3+1}$



When x gets bigger, it's like $f(x)$ gets closer to 1 even if $f(x)\neq 1$.

Similarly, if x gets smaller, $f(x)$ still gets closer to 1.

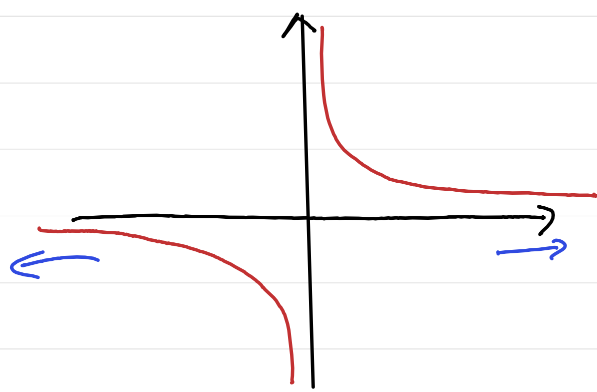
We describe "x gets bigger" as "x goes ∞" and "x gets smaller" as "x goes $-\infty$". In this case, we write

$$\lim_{x\to\infty} \frac{x^3-1}{x^3+1} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{x^3-1}{x^3+1} = 1$$

And we call $y=1$ as a <u>horizontal asymptote</u>.

<u>Examples</u> $\lim_{x\to\infty} \frac{1}{x} = 0$ since when x gets bigger, $\frac{1}{x}$ gets smaller but never 0.

Similarly, $\lim_{x\to-\infty} \frac{1}{x} = 0$.



In general, we have

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 = \lim_{x\to-\infty} \frac{1}{x^r}$$

for $r > 0$.

Example: Find the horizontal and vertical asymptotes of $f(x) = \dfrac{2x^2+5}{3x^2-3}$.

Solution: For horizontal asymptotes, we should check limits at $\pm\infty$. For vertical asymptotes, we should check where the limit is infinity.

$$\lim_{x\to\infty} \frac{2x^2+5}{3x^2-3} = \lim_{x\to\infty} \frac{2+\frac{5}{x^2}}{3-\frac{3}{x^2}} = \frac{\lim_{x\to\infty} 2+\frac{5}{x^2}}{\lim_{x\to\infty} 3-\frac{3}{x^2}} = \frac{2+0}{3-0} = \frac{2}{3}$$

divide both nominator and denominator by the highest power of x in denominator

both limit exist, use ratio rule

Similarly, $\lim_{x\to-\infty} \dfrac{2x^2+5}{3x^2-3} = \dfrac{2}{3}$. So $\boxed{y = \frac{2}{3}}$ is a horizontal asymptote.

Since the function is undefined at ± 1.
let's check these points.

$$\lim_{x \to 1^+} \frac{2x^2+5}{3x^2-3} = \infty$$

since if $x > 1$ and very close to 1, then the nominator is positive but the denominator is positive and getting smaller. The ratio increases.

We can decide $\boxed{x=1}$ is a vertical asymptote.

$$\lim_{x \to -1^-} \frac{2x^2+5}{3x^2-3} = \infty$$ by a similar reason.

We can decide $\boxed{x=-1}$ is another vertical asymptote.

Remarks We can have infinite limit at infinity. In other words, for some f, we can have

$$\lim_{x \to \infty} f(x) = \infty.$$

Similarly, $\lim_{x \to -\infty} f(x) = \infty$, $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = -\infty$

are possible.

Example: $\lim_{x\to\infty} x^3 = \infty$, $\lim_{x\to -\infty} x^3 = -\infty$.

One can read "$\lim_{x\to\infty} f(x) = \infty$" as

"when x becomes large, $f(x)$ becomes large".

Example: Find $\lim_{x\to\infty} \frac{x^2+x}{3-x}$.

We get

$$\lim_{x\to\infty} \frac{x^2+x}{3-x} = \lim_{x\to\infty} \frac{x+1}{\frac{3}{x}-1} = -\infty .$$

divide both nominator and denominator by the highest power of x in denominator

since $\lim_{x\to\infty} x+1 = \infty$ and $\lim_{x\to\infty} \frac{3}{x} - 1 = -1$.

Examples Find $\lim\limits_{x \to -\infty} \dfrac{x-2}{x^2+1}$.

We get $\lim\limits_{x \to -\infty} \dfrac{x-2}{x^2+1} = \lim\limits_{x \to -\infty} \dfrac{\frac{1}{x} - \frac{2}{x^2}}{1 + \frac{1}{x^2}} = \dfrac{\lim\limits_{x \to -\infty} \frac{1}{x} - \frac{2}{x^2}}{\lim\limits_{x \to -\infty} 1 + \frac{1}{x^2}}$

divide both
nominator
and denominator
by x^2

$= \dfrac{0}{1} = 0$

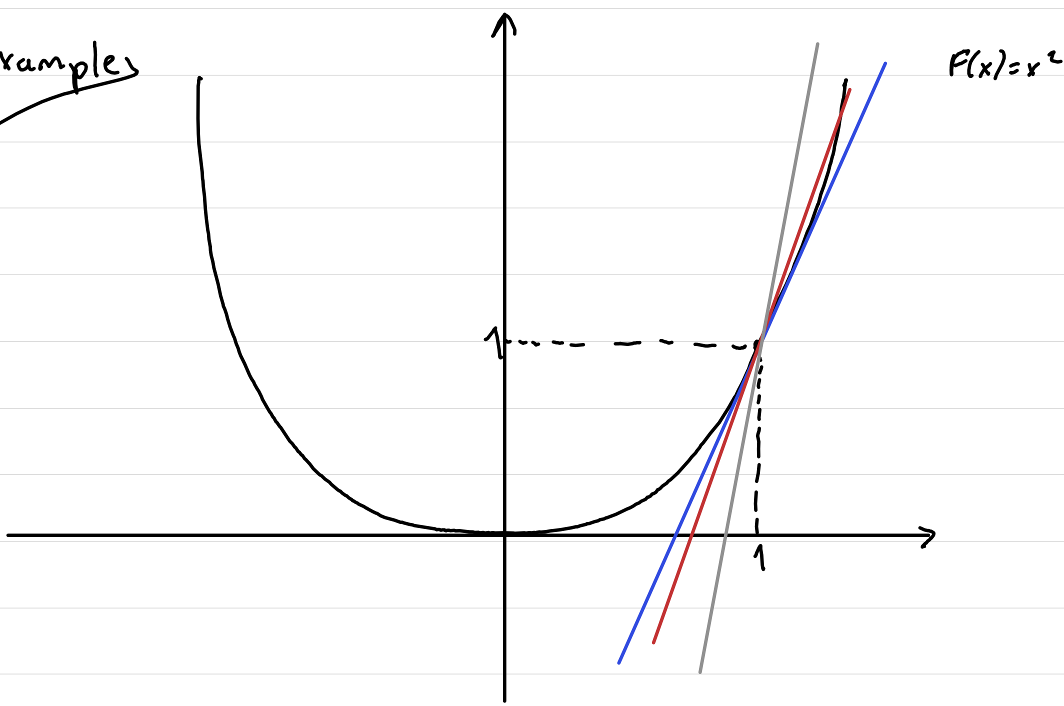
Derivatives and Rates of Changes

In this lecture, we will cover a special limit which we call it "derivative" later. In the end, derivative will be a rate of change, but we first give two intuitions about the special limit.

Tangent

A straight line touches a curve at a point, but if extended does not cross it at that point.

Examples



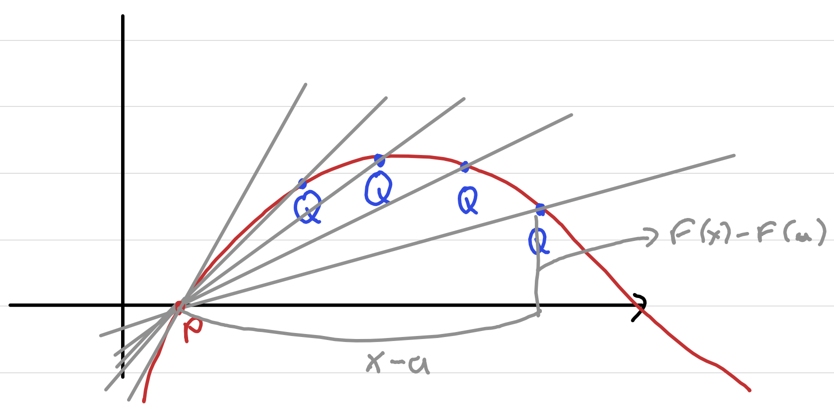
Blue line is the tangent line but red and gray ones are not. They are crossing the curve, namely, the graph of x^2.

How to find the tangent line?

Let P be the point $(a, f(a))$ in given curve and consider a nearby point Q $(x, f(x))$. The slope of the line passing through P and Q is given by

$$\frac{f(x)-f(a)}{x-a},$$

For different Q, we have different slope.



When x approaches a, namely Q approaches P, these lines approach the tangent line. Therefore, we have the following definition

Definition The tangent line to the curve $y=f(x)$ at the point $P=(a,f(a))$ is the line through P with the slope

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

if the limit exists.

Example Find the tangent line to $y=x^2$ at $(1,1)$.

Let $f(x)=x^2$. Then $$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}=\lim_{x\to 1}x+1=2.$$

So the tangent line is a line passing through $(1,1)$ with slope 2. The equation of the line is given by

$$y-1=2(x-1), \text{ namely, } \underline{\underline{y=2x-1}}.$$

Note that if a line passes through (a,b) and has slope m, then its equation is

$$y-b=m(x-a).$$

The limit $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ can be written also as $\boxed{\lim_{h\to 0}\frac{f(h+a)-f(a)}{h}}$ where $h=x-a$,

Velocity

We'll look at the same limit from another perspective. Suppose $f(x)$ describes the position of an object at given time x.

In the time interval from a to $a+h$, the object changes position from $f(a)$ to $f(a+h)$.

Then the average velocity of the object in given intervol is

$$\frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}.$$

When the intervol $[a, a+h]$ becomes shorter and shorter, we reach the velocity at the time $x=a$. So we define the <u>velocity</u> $v(a)$ at a by

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The <u>speed</u> at a is defined by $|v(a)|$.

Examples A particle moves along a straight line with equation of motion $s=f(t)=80t-6t^2$, where s is measured in meters and t in seconds. Find the velocity at $t=4$.

Solutions $\lim_{h \to 0} \frac{f(h+4)-f(4)}{h}$

$$= \lim_{h \to 0} \frac{80(h+4)-6(h+4)^2-(80(4)-6(4)^2)}{h}$$

$$= \lim_{h\to 0} \frac{80h + 320 - 6(h^2 + 8h + 16) - 320 + 6(16)}{h}$$

$$= \lim_{h\to 0} \frac{80h + \cancel{320} - 6h^2 - 48h - \cancel{96} - \cancel{320} + \cancel{96}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(80 - 6h - 48)}{\cancel{h}} = \lim_{h\to 0} 32 - 6h = 32.$$

For the same limit $\lim_{h\to 0} \frac{f(h+a) - f(a)}{h}$, we have two different meanings; The limit is

1) is the slope of tangent line to the curve at $x = a$

2) is the velocity of a particle at time $x = a$.

In both cases, the ratio $\frac{f(h+a) - f(a)}{h}$ represents a change of something. While the ratio is the average change of $f(x)$ with respect to x over $[a, a+h]$, the limit is the (instantaneous) rate of change at $x = a$.

Finally, we define the rate of changes as derivative.

Definition: The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(h+a) - f(a)}{h},$$

if the limit exists.

Example: Let $f(x) = 3x^2 - 4x + 1$. Find $f'(1)$.

$$f'(1) = \lim_{h \to 0} \frac{f(h+1) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3(h+1)^2 - 4(h+1) + 1 - (3(1)^2 - 4(1) + 1)}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + 6h + \cancel{3} - 4h - \cancel{4} + \cancel{1} - \cancel{3} + \cancel{4} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + 2h}{h} = \lim_{h \to 0} \frac{\cancel{h}(3h+2)}{\cancel{h}} = \lim_{h \to 0} 3h + 2$$

$$\boxed{= 2}$$

Derivative as a Function

Recall that last time we defined derivative;
For a function f, if the following limit exists

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

then we say f has a derivative at a, and the limit is the derivative itself. We denoted it by $f'(a)$.

Now, we will generalize the input a and talk about a general function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \qquad \varnothing$$

We call $f'(x)$ is the derivative function of f.
Note that if f has domain I, it <u>does not mean</u> that f' has also domain I.
$f'(x)$ is defined if and only if the limit $\varnothing$ exists.

Example: If $f'(x) = x^2 - x$, find $f'(x)$.

Solution: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\left((x+h)^2 - (x+h)\right) - (x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x} - h \cancel{- x^2} \cancel{+ x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h - 1 = 2x - 1.$$

So $f'(x) = 2x - 1$.

Example: If $f(x) = \sqrt{x}$, find $f'(x)$.

Solution: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(\cancel{x} + h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

So $f'(x) = \dfrac{1}{2\sqrt{x}}$.

Remarks For the derivative function, there are other well-known notations. All the following are the same: if $f(x)=y$, then

$$f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x).$$

For different purposes, we may use any of them. Circled ones are the most common ones.

Definitions A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an interval if it is differentiable at every number in the interval.

Examples $f(x)=x^2+x$ is a differentiable function (we showed before) and its derivative is $f'(x)=2x+1$.

! Note that not all function is differentiable.
We have a basic result:

<u>Theorem</u>: If f is differentiable at a, then f is continuous at a.

<u>Proof</u>: Omitted.

According to the theorem, we can say if f is not continuous at a point, then f cannot be differentiable at a.

<u>Example</u>: $f(x) = \begin{cases} x^2+1 & x \geq 0 \\ x & x < 0 \end{cases}$ is not continuous at 0, so f is not differentiable at 0.

Now, the main question is
How can a function fail to be differentiable?

The direct answer is that if $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ does not exist, then $f'(x)$ does not exist.

Example: We'll show that $f(x) = |x|$. differentiable everywhere except 0.

We want to calculate $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h}$.

Suppose $x > 0$. We can choose h small enough (since it approaches to 0) such that $x+h > 0$.

So we get $\lim\limits_{h \to 0} \frac{|x+h| - |x|}{h} = \lim\limits_{h \to 0} \frac{\cancel{x}+h - \cancel{x}}{h}$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 1 = 1.$$

So $f(x)$ is differentiable on $(0, \infty)$.

Suppose $x < 0$. Again we can choose h small enough such that $x+h < 0$, so we get

$$\lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-x} - h + \cancel{x}}{h} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}} = \lim_{h \to 0} -1 = -1.$$

So f is differentiable on $(-\infty, 0)$.
What happens at 0?

If exists, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

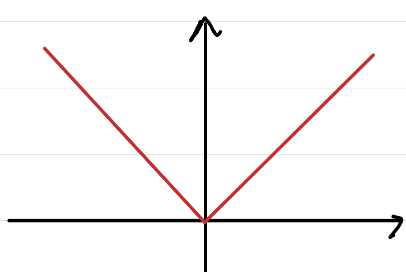
$$= \lim_{h\to 0} \frac{|h|}{h}. \quad \emptyset$$

So we need to find $\lim_{h\to 0} \frac{|h|}{h}$. However,

$\lim_{h\to 0^-} \frac{|h|}{h} = -1$ and $\lim_{h\to 0^+} \frac{|h|}{h} = 1$. So the

limit $\emptyset$ does not exist. So $f(x) = |x|$ is not differentiable at 0.

Consider the graph of $|x|$



at 0, the graph has a corner. In such a case $f'(x)$ does not exist.

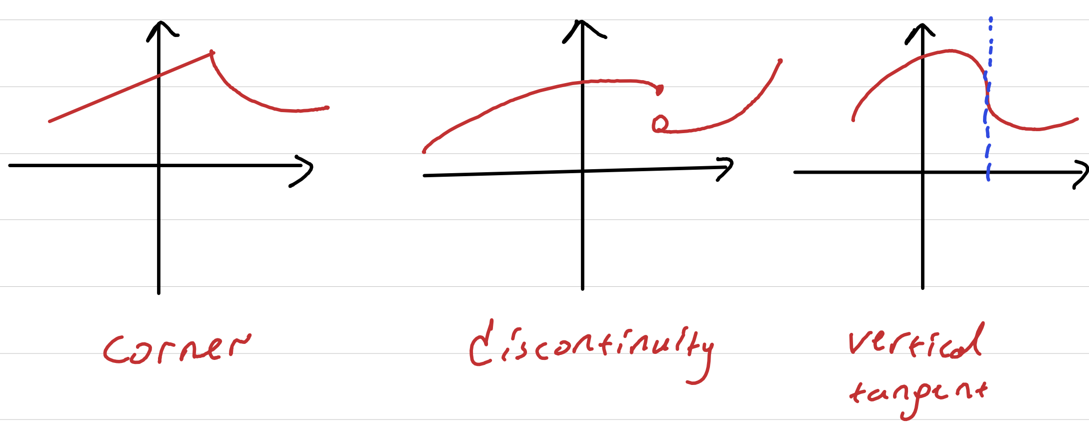
The problem at a corner point is that, like in the example, the left and the right limits are different.

Another possibility for being <u>not differentiable</u> is that we may have vertical tangent line when $x=a$. Namely, f is continuous at a, but

$$\lim_{x\to a} |f'(x)| = \infty.$$

The derivative is not a finite number.

Therefore, we have the following three cases for failing to be differentiable:



Note that we can repeat taking derivative and talk about second derivative, third derivative, etc.

$$f''(x),\ f'''(x),\ \ldots,\ f^{(n)}(x).$$

Be careful!!!

$$f^{(n)}(x) \quad \text{and} \quad f^{n}(x)$$

are not the same. The first is the n^{th} derivative, the second is n times product, namely $\underbrace{f(x)\cdot f(x)\cdot \ldots \cdot f(x)}_{n \text{ times}}$.

Example: Find $f''(x)$ for $f(x) = x^2 + x$.

Solution: Recall the first derivative $f'(x) = 2x + 1$. To find $f''(x)$, we'll take derivative of $f'(x)$.

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{(2(x+h)+1) - (2x+1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2h + \cancel{1} - \cancel{2x} - \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}} = \lim_{h\to 0} 2 = 2.$$

So $f''(x) = 2$.

Exercise. Show that for $f(x) = x^2 + x$,
$f'''(x) = 0$.

Derivatives of some essential functions

① $\boxed{\text{Constant Function}}$

If $f(x) = c$ for all x, then $f'(x) = 0$.

In other words

$$\frac{d}{dx}(c) = 0.$$

2 Power Rule

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

Examples

- $\frac{d}{dx}(x^3) = 3x^2$

- $\frac{d}{dx}\left(x^{\frac{5}{2}}\right) = \frac{5}{2}\,x^{\frac{5}{2}-1} = \frac{5}{2}\,x^{\frac{3}{2}}$

- $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}\,x^{\frac{1}{2}-1} = \frac{1}{2}\,x^{-1/2}$

$$= \frac{1}{2} \cdot \frac{1}{x^{1/2}}$$

$$= \frac{1}{2\sqrt{x}}$$

etc.

③ $\boxed{\text{Scalar multiple rule:}}$

$$\frac{d}{dx}\left(b\,f(x)\right) = b\left(\frac{d}{dx}f(x)\right)$$

<u>Example</u> • $\frac{d}{dx}\left(3x^2\right) = 3\left(\frac{d}{dx}x^2\right) = 3(2x) = 6x$

• $\frac{d}{dx}\left(4x^3\right) = 4\left(\frac{d}{dx}x^3\right) = 4\left(3x^2\right) = 12x^2$

④ $\boxed{\text{Sum / Difference rule:}}$

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x),$$

in other words;

$$(f \pm g)' = f' \pm g'.$$

Examples

- $$\frac{d}{dx}\left(x^3+x^2\right)$$

$$=\frac{d}{dx}x^3+\frac{d}{dx}x^2=3x^2+2x$$

- $$\frac{d}{dx}\left(3x^4-x\right)$$

$$=\frac{d}{dx}(3x^4)-\frac{d}{dx}(x)=3\left(\frac{d}{dx}x^4\right)-(1)$$

$$=3(4x^3)-1$$

$$=12x^3-1$$

! Now, we have everything needed to take the derivative of a polynomial.

Examples

- $F(x)=4x^2+3x-5$

Then $$\frac{d}{dx}(F(x)=4\frac{d}{dx}(x^2)+3\frac{d}{dx}(x)-\frac{d}{dx}(5)$$

$$=4(2x)+3(1)-(0)=8x+3.$$

- $P(x) = x^5 - 3x^3 + x^2$, Then

$$P'(x) = 5x^4 - 9x^2 + 2x$$

- $Q(x) = 10x^4 + 8x^2 - 7x + 4$, Then

$$Q'(x) = 40x^3 + 16x - 7$$

You can study on your examples.

(5) $\boxed{\text{Exponential Function}}$

Let $f(x) = b^x$ where b is a positive constant.

Then $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} = b^x \left(\lim_{h \to 0} \frac{b^h - 1}{h} \right).$$

Actually, if the limit exists, we know

$$\lim_{h \to 0} \frac{b^h - 1}{h} = f'(0) \quad \text{here.}$$

So for $f(x) = b^x$, we have

$$f'(x) = f'(0)\, b^x . \qquad \varnothing$$

! We'll learn to compute $\lim_{h \to 0} \frac{b^h - 1}{h}$ LATER.

Recall the number "e". It is designed to have $f'(0) = 1$ in $\varnothing$. In other words, e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 .$$

Therefore $\boxed{\frac{d}{dx}(e^x) = e^x}$. In other words,

the derivative of e^x is itself.

Examples $\dfrac{d}{dx}(3e^x + x^5)$

$$= 3\frac{d}{dx}(e^x) + \frac{d}{dx}(x^5)$$

$$= 3e^x + 5x^4.$$

⑥ **Product rule:** If f, g are both differentiable, then

$$\frac{d}{dx}\Big(f(x)\,g(x)\Big) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x)).$$

In other words, $(fg)' = fg' + gf'$.

! Its proof is omitted.

Examples • Let $H(x) = x^2 e^x$. Then by product rule:

$$H'(x) = \frac{d}{dx}(x^2 e^x) = x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2)$$

$$= x^2 e^x + e^2(2x) = x^2e^x + 2xe^x$$

- Let $Q(x) = \sqrt{x}\,(2+3x)$. Then by product rule:

$$Q'(x) = \sqrt{x}\,\frac{d}{dx}(2+3x) + (2+3x)\frac{d}{dx}(\sqrt{x})$$

$$= \sqrt{x}\,(3) + (2+3x)\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{3\sqrt{x}}{\underset{(2\sqrt{x})}{1}} + \frac{2+3x}{2\sqrt{x}}$$

$$= \frac{6x}{2\sqrt{x}} + \frac{2+3x}{2\sqrt{x}} = \frac{2+9x}{2\sqrt{x}}$$

7 $\boxed{\text{Quotient Rule}}$: If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}.$$

In other words,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Examples • Let $h(x) = \dfrac{e^x}{x^2+1}$. Then by quotient rule:

$$h'(x) = \frac{(x^2+1)\frac{d}{dx}(e^x) - e^x \frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)e^x - e^x(2x)}{x^4+2x^2+1}$$

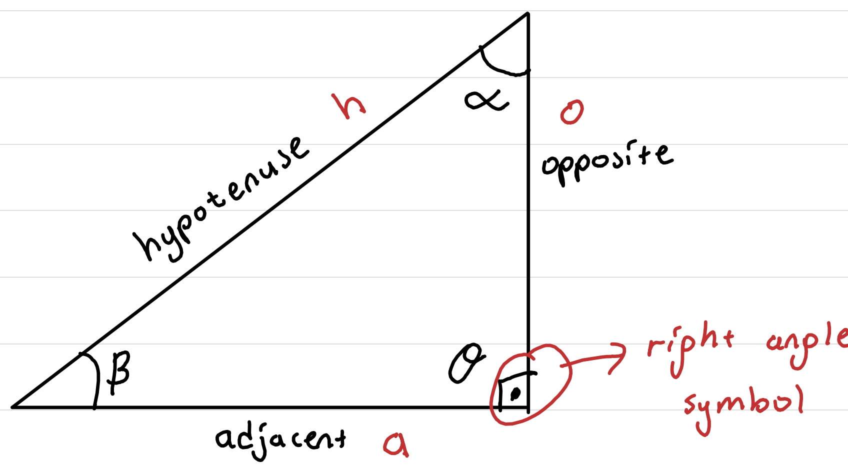
• Let $u(t) = \dfrac{t^3}{e^t+1}$

$$u'(t) = \frac{(e^t+1)\frac{d}{dt}(t^3) - t^3\frac{d}{dt}(e^t+1)}{(e^t+1)^2}$$

$$= \frac{(e^t+1)(3t^2) - t^3(e^t)}{(e^t+1)^2}.$$

Trigonometry (Digression)

It is the study of the relationship between the angles and sides of triangles, particularly right-angled triangles.



α, β, θ are angles.

For the sides of the triangle, we have the formula:

$$h^2 = a^2 + o^2$$

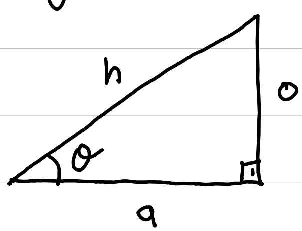
For angles, we have the formula

$$\alpha + \beta + \theta = 180^\circ$$

Angles can be measured in degrees or in radians The relationship between degree and radian is

$$\pi \text{ rad} = 180°,$$

Trigonometric Functions

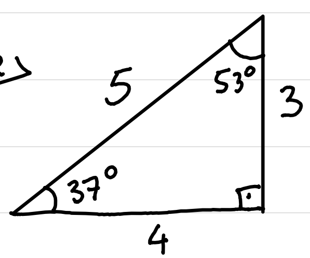


$$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a} \qquad \cot\theta = \frac{a}{o}$$

$$\sec\theta = \frac{h}{a} \qquad \csc\theta = \frac{h}{o}$$

Examples

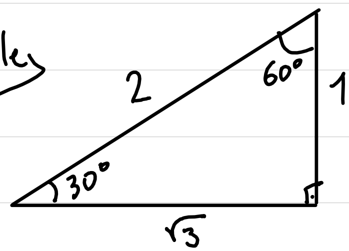


$$\sin 37° = \frac{3}{5} \qquad \tan 53° = \frac{4}{3}$$

$$\sec 37° = \frac{5}{4}$$

$$\cos 53° = \frac{3}{5}$$

Examples

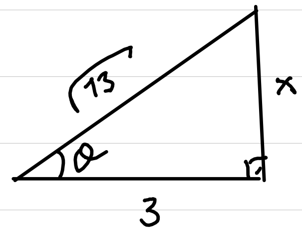


$\sin 30° = \sin \frac{\pi}{6} = \frac{1}{2}$ $\csc 60° = \csc \frac{\pi}{3} = \frac{2}{\sqrt{3}}$

$\cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}$, find the other values as an exercise.

Examples Using a trigonometric value, we can depict the corresponded triangle,

If $\cos \theta = \frac{3}{\sqrt{13}}$, we have



by the formula we get

$x^2 + 3^2 = \sqrt{13}^2$

$x^2 + 9 = 13$

$x^2 = 4$

$\boxed{x = 2}$

Remarks In Week 2 notes, there is a file about major trigonometric identities. Some famous ones are

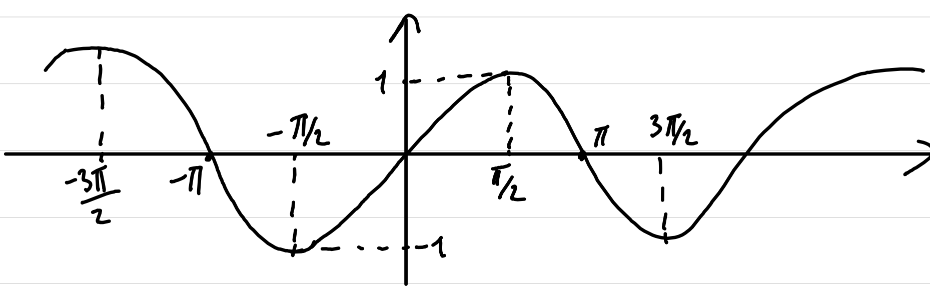
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1,$$

Properties of trigonometric functions

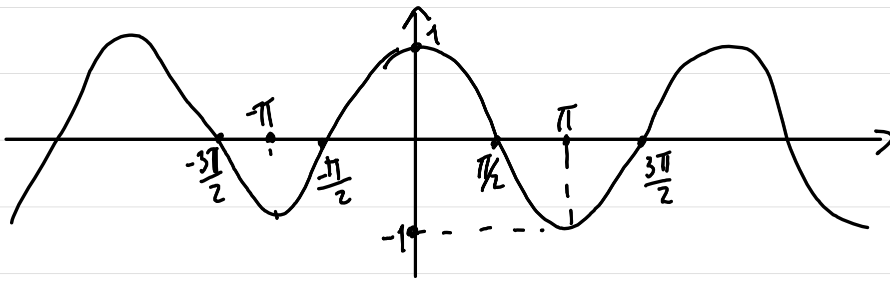
- $\sin\theta$ has domain $\mathbb{R}$ and range $[-1, 1]$



$\sin\theta$ is an odd function

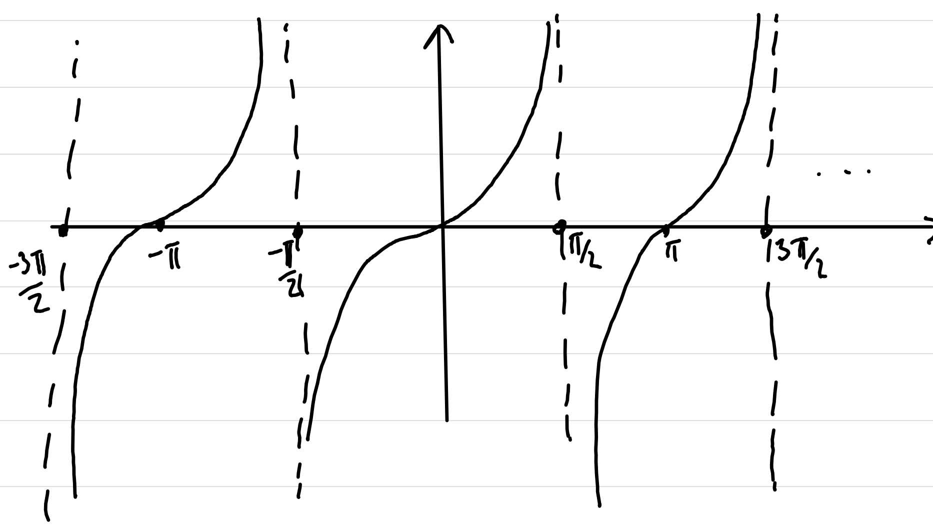
$$\sin(-\theta) = -\sin\theta$$

- $\cos\theta$ has domain $\mathbb{R}$ and range $[-1,1]$.

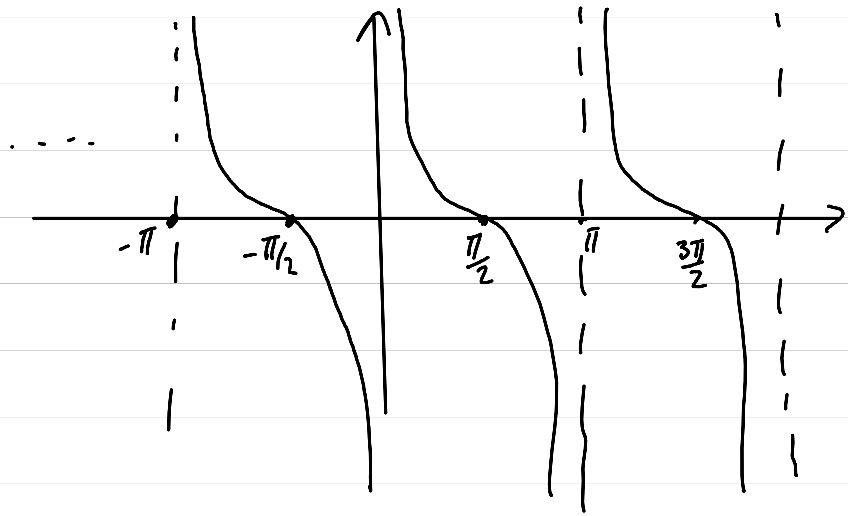


$\cos\theta$ is an even function
$$\cos(-\theta) = \cos\theta$$

- $\tan\theta$ is defined everwhere except for $\theta = \frac{\pi}{2} + n\pi$ for any integer n. Its range is $\mathbb{R}$.



• $\cot\theta$ is defined everywhere except for $\theta = n\pi$ for any integer n. Its range is $\mathbb{R}$



! Please read the notes about trigonometry for the details.

Using some trigonometric identities, we can solve equations.

Examples Find all values of x in the interval $[0, 2\pi]$ that satisfy

$$2\cos x + \sin 2x = 0.$$

Solution: We have $\sin 2x = 2\sin x \cos x$.

$2\cos x + \sin 2x = 0$ means

$$2\cos x + 2\sin x \cos x = 0$$

$$2\cos x\,(1 + \sin x) = 0$$

So either $\cos x = 0$ or $1 + \sin x = 0$.

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

it means $\sin x = -1$,

so $x = \frac{3\pi}{2}$

Therefore, $2\cos x + \sin 2x = 0$ has two solutions, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, in $[0, 2\pi]$.

Derivative of trigonometric Functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$

These need to be proved using the definition of derivative, but it's omitted.

In their proofs, the following limits are crucial.

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1} \qquad \boxed{\lim_{x \to 0} \frac{\cos x - 1}{x} = 0}$$

The other derivatives can be obtained by derivative rules.

Examples $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x (\cos x) - \sin x (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Examples $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

$$= \frac{\cos x \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \cdot \sec x$$

Exercise Use the quotient rule to calculate

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\qquad$ $\frac{d}{dx}(\csc x) = -\cot x \cdot \csc x$

Examples Find the derivative of $f(x) = x^2 \sin x$.

Use product rule:

$$f'(x) = (x^2)' \sin x + (x^2)(\sin x)'$$

$$= \boxed{2x \sin x + x^2 \cos x}$$

Remark: $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$ (⊗) is very useful in other limit calculations.

Example: Find $\lim_{x\to 0} \frac{\sin(5x)}{16x}$.

We want to have $\frac{\sin 5x}{5x}$ to use ⊗.

To get this, observe

$$\frac{\sin 5x}{16x} = \frac{5}{16} \cdot \frac{\sin 5x}{5x}.$$

So $\lim_{x\to 0} \frac{\sin 5x}{16x} = \lim_{x\to 0} \frac{5}{16} \cdot \frac{\sin 5x}{5x}$

$$= \frac{5}{16} \underbrace{\left(\lim_{x\to 0} \frac{\sin 5x}{5x}\right)}_{=1} = \frac{5}{16}$$

Example: Find $\lim_{x \to 0} \frac{\sin x}{\sin \pi x}$,

$$\lim_{x \to 0} \frac{\sin x}{\sin \pi x} = \lim_{x \to 0} \frac{\frac{\sin x}{\pi x}}{\frac{\sin \pi x}{\pi x}}$$

$$= \frac{\lim_{x \to 0} \frac{\sin x}{\pi x}}{\lim_{x \to 0} \frac{\sin \pi x}{\pi x}} = \frac{\frac{1}{\pi} \lim_{x \to 0} \frac{\sin x}{x}}{1}$$

$$= \frac{1}{\pi},$$

Thus, the trick is to make given expression involving "$\frac{\sin \theta}{\theta}$" parts when $\theta \to 0$.

$\boxed{\text{Chain Rule}}$

Recall that if $f: A \to B$ and $g: C \to A$ are two functions, we can take their composition $f \circ g: C \to B$. We'll learn how to take the derivative of composition of functions.

Chain rule: If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and it is given by

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}.$$

Another notation: If $y = f(u)$ and $u = g(x)$, we also write

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Examples Let $F(x) = \sqrt{x^2+1}$,

Then $F(x)$ is the composition of $f(x) = \sqrt{x}$ and $g(x) = x^2+1$. In other words,

$$F(x) = f \circ g\,(x)$$

By the chain rule, we have

$$F'(x) = f'(g(x)) \cdot g'(x).$$

We know $f'(x) = \dfrac{1}{2\sqrt{x}}$ and $g'(x) = 2x$. So

$$F'(x) = \frac{1}{\cancel{2}\sqrt{x+1}} \cdot \cancel{2}x$$

$$\boxed{F'(x) = \frac{x}{\sqrt{x+1}}}$$

! While taking the derivative of $f \circ g\,(x)$:

1. take derivative of $f(x)$
2. evaluate $f'(x)$ at $g(x)$
3. take derivative of $g(x)$
4. multiply $f'(g(x))$ and $g'(x)$

Example: We'll use chain rule to take the derivative of b^x.

Let $f(x) = b^x$.

Since e^t and $\ln t$ are inverses of each other, we have

$$\boxed{e^{\ln b} = b}$$

(Recall $g \circ g^{-1}(x) = x$ and $g^{-1} \circ g(x) = x$)

So $f(x) = b^x = (e^{\ln b})^x = e^{(\ln b)x}$.

By chain rule, let $g(x) = (\ln b)x$ and $h(x) = e^x$, so $f(x) = h \circ g(x)$, we get

$$\begin{aligned} f'(x) &= h'(g(x))\, g'(x) \\ &= e^{(\ln b)x} (\ln b) \\ &= b^x (\ln b) \end{aligned}$$

So if $f(x) = b^x$, then $\boxed{f'(x) = b^x (\ln b)}$ //

Examples $F(x) = \sin^5(x)$.

This is composition of $f(x) = x^5$ and $g(x) = \sin x$, namely,

$$F(x) = f \circ g(x),$$

So

$$F'(x) = f'(g(x))\, g'(x).$$

We know $f'(x) = 5x^4$ and $g'(x) = \cos x$.

So

$$\boxed{F'(x) = 5(\sin^4(x)) \cdot \cos(x)}.$$

Examples $F(x) = \sin(x^5)$

This is composition of $f(x) = \sin(x)$ and $g(x) = x^5$.

So $F'(x) = f'(g(x)) \cdot g'(x)$

$$\boxed{F'(x) = \cos(x^5) \cdot 5x^4}$$

! Be careful about the different compositions like in these two examples.

Example: $F(x) = (2x^3 + e^x)^{50}$.

This is composition of $f(x) = x^{50}$ and $g(x) = 2x^3 + e^x$. We have $f'(x) = 50x^{49}$ and $g'(x) = 6x^2 + e^x$. Thus,

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\boxed{F'(x) = 50(2x^3 + e^x)^{49} \cdot (6x^2 + e^x)}$$

Example: $F(x) = e^{\sin x}$.

This is a composition of $f(x) = e^x$ and $g(x) = \sin x$. Since $f'(x) = e^x$ and $g'(x) = \cos x$, we have

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\boxed{F'(x) = e^{\sin x} \cdot \cos x}$$

Examples Find the derivative of
$F(x) = \sin(\cos(x^2+1))$.

Here, we have composition of three functions $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = x^2+1$.

Note that $f'(x) = \cos x$, $g'(x) = -\sin x$, $h'(x) = 2x$,
Thus, applying chain rule twice, we get

$$
\begin{aligned}
F'(x) &= (f \circ g \circ h)'(x) \\
&= f'(g \circ h(x))\,(g \circ h)'(x) \\
&= f'(g \circ h(x))\, g'(h(x)) \cdot h'(x)
\end{aligned}
$$

$$= \cos(\cos(x^2+1))\,(-\sin(x^2+1))\,(2x).$$

So
$$\boxed{F'(x) = -2x \cdot \cos(\cos(x^2+1)) \cdot (\sin(x^2+1))}$$

Examples Find the derivative of

$$F(x) = \frac{x^2 + e^{3x}}{\sin x}.$$

Here, we need sum rule, chain rule and quotient rule:

$$F'(x) = \frac{(x^2+e^{3x})' \sin x - (x^2+e^{3x})(\sin x)'}{\sin^2(x)}$$

quotient rule

$$= \frac{(2x + 3e^{3x}) \sin x - (x^2+e^{3x})(\cos x)}{\sin^2(x)}$$

Sum rule and chain rule.

! We used chain rule in e^{3x}. It is composition of $f(x) = e^x$ and $p(x) = 3x$.

$$(e^{3x})' = f'(p(x)) \cdot p'(x)$$
$$= e^{3x} \cdot 3 = \underline{\underline{3e^{3x}}}$$

Summary for derivative topics so far:

- Definition of $f'(x)$

  if $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists,

  then it's $f'(x)$.

- Basic rules
  - $(f \pm g)' = f' \pm g'$
  - $(x^n)' = n\, x^{n-1}$
  - $(a)' = 0$ where a constant
  - $(e^x)' = e^x$

- Advanced rules
  - $(fg)' = f'g + fg'$
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

- Trig rules
  - $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$
  - the others are by quotient rules

- Chain rule
  - $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

- $(b^x)' = (\ln b)\, b^x$
- $(f^n(x))' = n f^{n-1}(x) \cdot f'(x)$

As you might observe the derivative of logarithmic function is missing. We will cover another method first, and use the method to find the derivative of logarithms.

Implicit Differentiation

So far, we learned how to differentiate the expressions like

$$y = x^2 + 5 \quad , \; y = x\sin x \; , \text{ etc.}$$

What about the expressions like

$$x^2 + y^2 = 25 \quad , \quad x^3 + y^3 = 6xy \quad ?$$

How we can find $\frac{dy}{dx}$ in such cases?

One way, We can solve the equation.

For example $x^2 + y^2 = 25$ gives

$y^2 = 25 - x^2$, so

$y = \pm\sqrt{25 - x^2}$.

But for the equation $x^3 + y^3 = 6xy$, it's not easy.

Implicit diffentiation is another and more efficient way. It consists of differentiating both sides.

Examples $x^2 + y^2 = 25$, find $\frac{dy}{dx}$

Solutions

- First take the derivative of both sides:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

- Apply the sum, constant and power rule:

$$2x + \frac{d}{dx}(y^2) = 0$$

- Apply the chain rule in the second part:

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

- We have

$$2x + 2y\frac{dy}{dx} = 0.$$

- Leave $\frac{dy}{dx}$ alone

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

- Finally, we get $\frac{dy}{dx} = -\frac{x}{y}$.

Example: Find an equation of the tangent to the curve $x^2+y^2=25$ at $(3,4)$.

Solution: We already found $\frac{dy}{dx} = -\frac{x}{y}$.

So if $x=3$, $y=4$, then $\frac{dy}{dx} = -\frac{3}{4}$.

So the tangent line equation is

$$(y-4) = -\frac{3}{4}(x-3)$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 4$$

$$y = \frac{-3x}{4} + \frac{25}{4} = \frac{-3x+25}{4}$$

So $4y = -3x + 25$, namely

$$\underline{\underline{4y + 3x = 25}}.$$

NOTE that $\frac{dy}{dx}$ and y' are the same. We can use both notation!!!

~~Examples~~ Find the tangent line to the curve $x^3 + y^3 = 6xy$ at the point $(3,3)$

First we need the slope of the tangent, namely, y'. Using implicit differentiation:

$$(x^3 + y^3)' = (6xy)'$$

$$(x^3)' + (y^3)' = (6x)'y + (6x)y'$$

$$3x^2 + 3y^2 y' = 6y + 6xy'$$

$$3y^2 y' - 6xy' = 6y - 3x^2$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

At the point $(3,3)$, we get

$$y' = \frac{6 \cdot 3 - 3(3^2)}{3(3^2) - 6 \cdot 3}$$

$$= \frac{18 - 27}{27 - 18} = \frac{-9}{9} = -1.$$

So the tangent line equation is

$$(y-3) = (-1)(x-3)$$

$$y = -x + 6 \quad \text{or} \quad \underline{\underline{y + x = 6}}.$$

Example Find y' if $\sin(x+y) = y^2\cos x$ via implicit differentiation.

$$(\sin(x+y))' = (y^2\cos x)'$$

$$\cos(x+y)\cdot(x+y)' = (y^2)'\cos x + y^2(\cos x)'$$

$$\cos(x+y)(1+y') = 2yy'\cos x - y^2\sin x$$

$$\cos(x+y) + y'\cos(x+y) = 2yy'\cos x - y^2\sin x$$

$$y'\cos(x+y) - 2yy'\cos x = -y^2\sin x + \cos(x+y)$$

$$y'\left(\cos(x+y) - 2y\cos x\right) = -y^2\sin x + \cos(x+y)$$

$$\boxed{y' = \frac{-y^2\sin x + \cos(x+y)}{\cos(x+y) - 2y\cos x}}$$

Examples Find y'' if $x^4+y^4=16$.

Solution First find y'.

$$(x^4+y^4)'=(16)'$$

$$4x^3+4y^3y'=0$$

$$4y^3y'=-4x^3$$
$$y^3y'=-x^3 \longrightarrow \boxed{y'=\frac{-x^3}{y^3}}$$

$$y''=\left(\frac{-x^3}{y^3}\right)'=\frac{(-x^3)'y^3-(-x^3)(y^3)'}{y^6}$$

$$=\frac{-3x^2y^3+x^3\,3y^2y'}{y^6}$$

$$=\frac{-3x^2y^3+3x^3y^2\left(\frac{-x^3}{y^3}\right)}{y^6}$$

If we continue to simplify, we get

$$y'' = \frac{-3x^2y^3 - \frac{3x^6}{y}}{y^6}$$

$$= \frac{\frac{-3x^2y^4 - 3x^6}{y}}{y^6}$$

$$= \frac{-3x^2y^4 - 3x^6}{y^7}$$

$$= \frac{-3x^2(y^4+x^4)}{y^7} = \frac{-3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$

Ex: Find the tangent line equation to the curve $x^2 - xy - y^2 = 1$ at $(2,1)$.

Solution: First find y' via implicit differentiation.

$$(x^2 - xy - y^2)' = (1)'$$

$$2x - (y + xy') - 2yy' = 0$$

$$2x - y - xy' - 2yy' = 0$$

$$y'(-x - 2y) = y - 2x$$

$$y' = \frac{y - 2x}{-x - 2y}$$

at $(2,1)$, we have $y' = \dfrac{1-4}{-2-2} = \dfrac{-3}{-4} = \dfrac{3}{4}$.

The equation is

$$y - 1 = \frac{3}{4}(x - 2),$$

Now, using implicit differentiation, we'll find the derivative of $\ln x$.

Suppose $y = \log_b x$ for $b > 0$.

It means $b^y = x$. Taking the derivative both sides via implicit differentiation

$$(b^y)' = (x)'$$

$$b^y (\ln b)\, y' = 1$$

$$y' = \frac{1}{b^y (\ln b)} = \frac{1}{x (\ln b)}.$$

So $\boxed{(\log_b x)' = \dfrac{1}{x\,(\ln b)}}$.

Examples $(\ln x)' = \dfrac{1}{x(\ln e)} = \dfrac{1}{x}$ since $\ln e = 1$.

Examples Differentiate $y = \ln(x^3+1)$.

We have a composition of $\ln x$ and x^3+1, so we apply the chain rule.

$$y' = \ln'(x^3+1)\,(x^3+1)'$$
$$= \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1}$$

recall $\ln'(A) = \frac{1}{A}$

In general, we have $\left[\ln(g(x))\right]' = \dfrac{g'(x)}{g(x)}$.

Examples Find $f'(x)$ if $f(x) = \ln(\sin x)$.

$$f'(x) = \frac{(\sin x)'}{\sin x} = \frac{\cos x}{\sin x} = \cot x.$$

RELATED RATES

Note! This topic will be covered on next Monday (October 14) since there is no lecture on October 11. It's included in Exam 2 topic, so I recommend you to read before the lecture.

The topic is about problem solving methods. In a related rate problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity.

The procedure is to find an equation that relates the two quantities and then use the Chain rule to differentiate both sides with respect to time.

Examples Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \ cm^3/s$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solutions Let V be the volume of the balloon and let r be its radius, t be time

Given info is about

$$\frac{dV}{dt} = 100 .$$

recall diameter = 2 radius

We want to know $\dfrac{dr}{dt} = ?$ when $r = 25$

Here, we should recall the the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

We should take derivative with respect to t, namely $\frac{dV}{dt}$. But using the chain rule,

we get
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}.$$

Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$,

then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

It means $\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt} = \frac{1}{4\pi r^2} 100$.

If $r=25$, we get
$$\frac{dr}{dt} = \frac{1}{\cancel{4}\pi(25)^2} \cancelto{25}{100}$$
$$= \frac{\cancel{25}}{\pi (25)^{\cancel{2}}} = \frac{1}{25\pi}.$$

So $\frac{dr}{dt} = \frac{1}{25\pi}$.

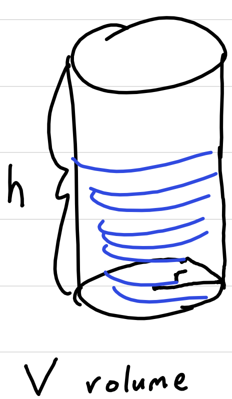
Examples If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.

Solutions The volume of a cube is

$$V = x^3.$$

$$\text{So} \quad \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = 3x^2 \frac{dx}{dt}.$$

Examples A cylindrical tank with radius 5m is being filled with water at a rate of $3\ m^3/min$. How fast is the height of the water increasing?



$$V = \pi r^2 h = \pi (5^2) h = 25\pi h.$$

$$\text{So} \quad \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = (25\pi)\frac{dh}{dt}.$$

Since $\frac{dV}{dt} = 3$, we get $25\pi \frac{dh}{dt} = 3$, so $\frac{dh}{dt} = \frac{3}{25\pi}$.

Examples A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4,2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

Solutions Take derivative of $xy = 8$ with respect to t. We get

$$\frac{d}{dt}(xy) = \frac{d}{dt}(8)$$

$$\frac{dx}{dt}y + x\frac{dy}{dt} = 0 \qquad (\varnothing)$$

Red highlighted sentence means $\frac{dy}{dt} = -3$ at point $(4,2)$. So if we use these in $\varnothing$, we get

$$\frac{dx}{dt}(2) + 4(-3) = 0$$

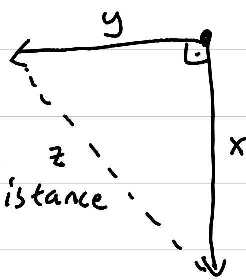
$$2\frac{dx}{dt} = 12$$

$$\boxed{\frac{dx}{dt} = 6}$$

Thus, the x-coordinate is <u>increasing</u> at a rate of 6 cm/s.

Example Two cars start moving from the same point. One (car y) travels south at 60 mi/h and the other (car x) travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Solution



We are given

$$\frac{dy}{dt} = 25, \quad \frac{dx}{dt} = 60.$$

Also $x^2 + y^2 = z^2$.

Thus, $\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(z^2)$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}. \qquad (\emptyset)$$

After 2 hours, $x = 2 \cdot 60 = 120$

$y = 2 \cdot 25 = 50$, so

$$z = \sqrt{x^2+y^2}$$
$$= \sqrt{120^2+50^2} = 130$$

So use all numbers in $\emptyset$, and get

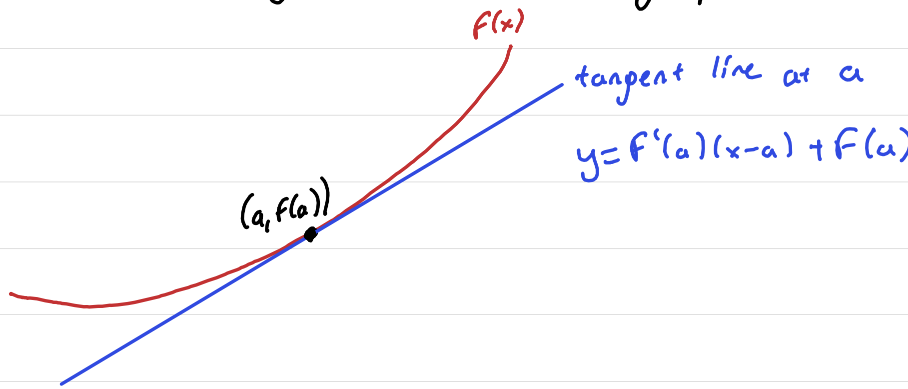
$$\cancel{2}(120)(60) + \cancel{2}(50)(25) = \cancel{2}(130)\frac{dz}{dt}$$

$$\frac{120(60)+50(25)}{130} = \frac{dz}{dt}$$

$$\boxed{65 = \frac{dz}{dt}}$$

Linear Approximations

Consider the following shape in a graph:



The curve lies very close its tangent near the point $(a, f(a))$. It might be easy to calculate $f(a)$, but not for nearby values. We can use this close line to approximate the actual values. In other words, we have

$$f(x) \approx \underbrace{f'(a)(x-a) + f(a)}$$ where x is near a.

This is called linear approximation of f at a.

We use it as a new function

$$L(x) = f'(a)(x-a) + f(a)$$

which is called linearization of f at a.

Remark: This is NOT something new. Basically, we compute the tangent line at a given point $(a, f(a))$ and use it as a linearization.

Example: Find the linearization $L(x)$ of the function $f(x) = x^3 - x^2 + 3$ at $a = -2$.

Solution: To get the point $(a, f(a))$, first find $f(a)$.

$$f(-2) = (-2)^3 - (-2)^2 + 3 = -8 - 4 + 3 = -9.$$

To get $f'(a)$, first find the derivative $f'(x)$.

$f'(x) = 3x^2 - 2x$, so

$$f'(-2) = 3(-2)^2 - 2(-2) = 3 \cdot 4 + 4 = 16.$$

Thus, the linearization is

$$L(x) = f'(-2)(x-(-2)) + f(-2)$$

$$\boxed{L(x) = 16(x+2) - 9}.$$

Example: Find the linearization of
$f(x) = \ln(1+x)$ at $x = 1$.

Solution: Let's repeat what we need:

$$L(x) = f'(1)(x-1) + f(1).$$

So we should find $f(1)$ and $f'(1)$.

$f(1) = \ln(1+1) = \ln 2$.

$f'(x) = \dfrac{1}{1+x}$, so $f'(1) = \dfrac{1}{2}$.

Therefore,

$$L(x) = \frac{1}{2}(x-1) + \ln 2.$$

How we can use linear approximation in practice?

Suppose we want to compute $\sqrt{3.98}$ which is very hard to achieve by hand. But we can find an approximate value.

Consider a close integer to 3.98 which is 4. And the function we look is $f(x)=\sqrt{x}$. So we can use linearization of $\sqrt{x}$ at 4 to approximate $\sqrt{3.98}$.

$$L(x) = f'(4)(x-4) + f(4) \quad \text{where } f(x)=\sqrt{x}.$$

So $f(4)=\sqrt{4}=2$ and $f'(x)=\dfrac{1}{2\sqrt{x}}$, so $f'(4)=\dfrac{1}{4}$.

Thus, $L(x)=\dfrac{1}{4}(x-4)+2$

$$= \frac{x}{4} - 1 + 2 = \frac{x}{4} + 1.$$

Then we have $\sqrt{3.98} \approx L(3.98) = \dfrac{3.98}{4} + 1$

$$= 0.995 + 1 = \boxed{1.995}$$

Example: Estimate $\cos 29^\circ$.

We know $\cos 30^\circ = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use linearization of $\cos x$ at 30°.

$(\cos x)' = -\sin x$, so $-\sin 30^\circ = -\frac{1}{2}$.

$$L(x) = -\frac{1}{2}(x - 30) + \frac{\sqrt{3}}{2}$$

So $\cos 29 \approx L(29) = -\frac{1}{2}(29 - 30) + \frac{\sqrt{3}}{2}$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}.$$

Exercise: Estimate $(1.999)^4$ using linearization of $f(x) = x^4$ at 2.

Maximum and Minimum Values

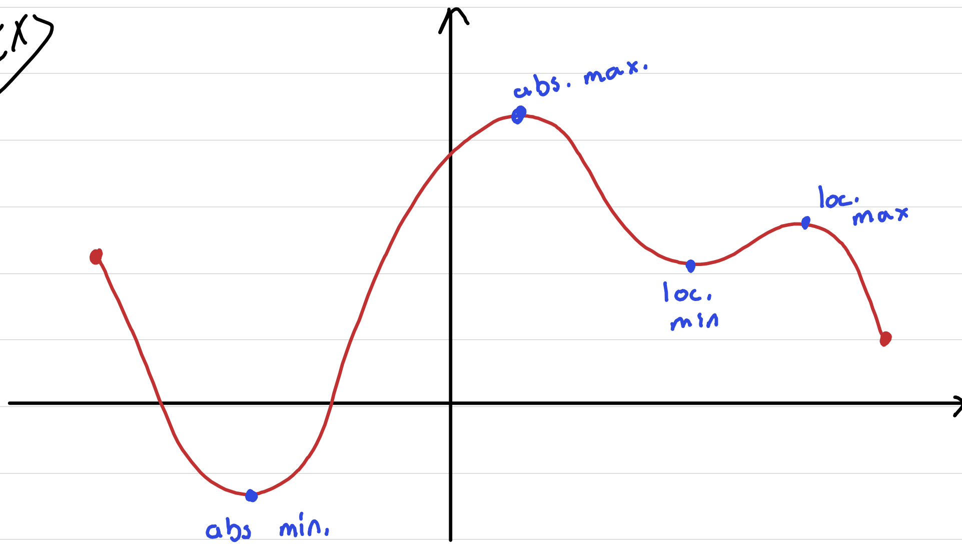
In this lecture, we'll learn what do maximum value and minimum value mean? and how do these relate with the derivative?
Why do we care max/min values? Because they are related to optimization problems. We might want to minimize loss and to maximize gain. Therefore, we should learn how to achieve these.

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is

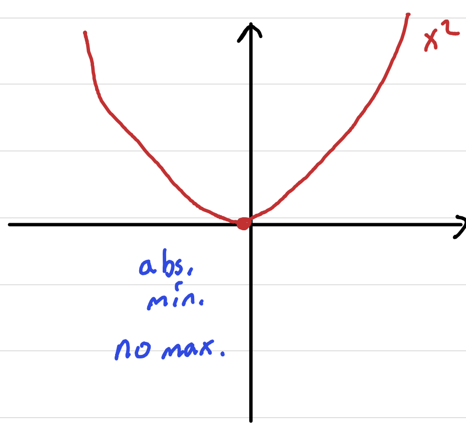
- the absolute max. value of f in D if
  $f(c) \geq f(x)$ for all x in D,
- the absolute min. value of f in D if
  $f(c) \leq f(x)$ for all x in D,
- a local max. value if
  $f(c) \geq f(x)$ for x near c,
- a local min. value if
  $f(c) \leq f(x)$ for x near c.

It's better to see these in an example:

Ex)

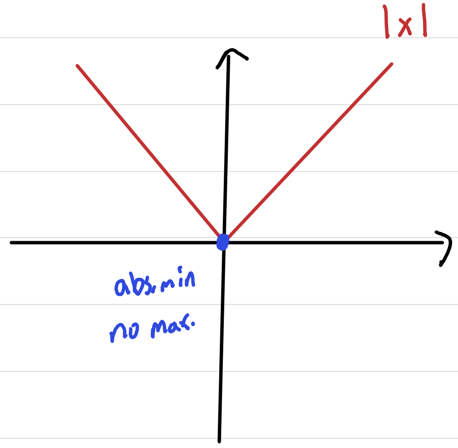


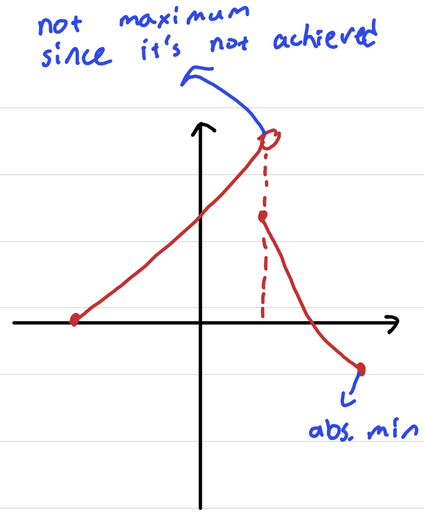
Ex)



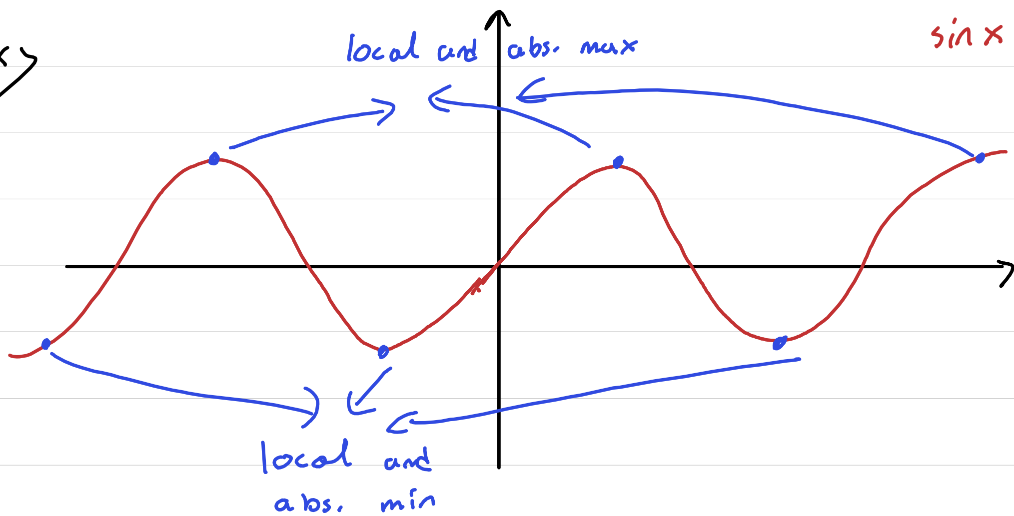
! Note that each abs. max./min is also a local max./min.

Ex)





Ex)



The following two theorems declare when max./min. values exist and what is the relationship between these and the derivative.

(1) Extreme Value Theorem

If f is continuous on $[a,b]$, then f attains an abs. max. $f(c)$ and an abs. min. $f(d)$ at some numbers c, d in $[a,b]$.

(2) Fermat's Theorem

If f has a local minimum or maximum at c, and $f'(c)$ exists, then $f'(c)=0$.

Therefore, in a continuous functions, we can find the possible max. min. values by looking at the points where the derivative is zero. Although, at those max. min points, the derivative is zero, this does not mean, however, that if the derivative is zero, then there should be a max. or min at that point.

Ex) $f(x)=x^3$ has $f'(0)=3(0^2)=0$ but x^3 has no max. or min.

Ex) $f(x)=|x|$ has a minimum at 0, but $f'(0)$ does not exist.

We'll provide a method to find abs max./min. values of a continuous function using the following new definition.

Definition: A critical number of a function f is a number c in the domain of f such that

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ DNE}.$$

Ex: Find the critical numbers of

$$f(x) = x^3 + 6x^2 - 15x.$$

Consider $f'(x) = 3x^2 + 12x - 15$. This is defined everywhere. So we should look at when $f'(x) = 0$.

$$3x^2 + 12x - 15 = 3(x^2 + 4x - 5)$$
$$= 3(x+5)(x-1) = 0 \quad \text{means}$$
$$x = -5 \quad \text{or} \quad x = 1$$

So $f'(-5) = 0$ and $f'(1) = 0$.

The critical numbers are -5 and 1.

Examples Find the critical numbers of

$f(x) = x^{-2} \ln x$ on $(0, \infty)$

$$f'(x) = -2x^{-3} \ln x + x^{-2}\left(\frac{1}{x}\right)$$

$$= -2x^{-3} \ln x + x^{-3}$$

$$= \frac{-2\ln x}{x^3} + \frac{1}{x^3}$$

$$= \frac{-2\ln x + 1}{x^3}$$

$f'(0)$ DNE and if $f'(x) = 0$, then it means $\frac{-2\ln x + 1}{x^3} = 0$, so $-2\ln x + 1 = 0$

$$1 = 2\ln x$$

$$\frac{1}{2} = \ln x$$

$$\sqrt{e} = e^{1/2} = x$$

We don't count 0 since it's not in domain. So the only critical number is $\sqrt{e}$.

Example: Find the critical numbers of

$$f(x) = 2\cos x + \sin^2 x.$$

$$f'(x) = -2\sin x + 2\sin x\cos x$$
$$= 2\sin x\,(-1+\cos x)$$

This is defined everywhere. Let's check when $f'(x) = 0$. It means

$$2\sin x\,(-1+\cos x) = 0,$$ so

$\sin x = 0$ or $-1+\cos x = 0$

$\cos x = 1$

$\sin x = 0 \downarrow$ $x = 2\pi n$ or $x = \pi + 2\pi n$ for n integer

$\cos x = 1 \downarrow$ $x = 2\pi n$ for n integer

So for all n integer, $2\pi n$ and $\pi + 2\pi n$ are critical numbers.

Now, let's describe the method to find abs. max./min values.

$\boxed{\text{METHOD}}$ Let $f(x)$ be continuous on $[a,b]$.

1. Find the values of f at the critical numbers of f in (a,b).
2. Find $f(a)$ and $f(b)$.
3. Compare the numbers, the largest one will be abs. max., the smallest one will be abs. min.

Ex) Find the abs. min and/or abs. max of

$$f(x) = x^3 + 6x^2 - 15x$$

on $[-7, 7]$.

Answer) 1. We alread found that $-5, 1$ are critical numbers.

$$f(-5) = (-5)^3 + 6(-5)^2 - 15(-5)$$
$$= -125 + 150 + 75 = 100$$

$F(1) = 1^3 + 6(1)^2 - 15 \cdot 1 \quad = 1 + 6 - 15 = -8$

② $f(-7) = (-7)^3 + 6(-7)^2 - 15(-7)$

$= -343 + 6 \cdot 49 + 105$

$= -343 + 294 + 105$

$= 56$

$f(7) = 7^3 + 6(7)^2 - 15(7)$

$= 532$

I omitted the steps. :)

③ So we have

$F(-5) = 100$

$F(1) = -8 \longrightarrow$ Abs. min.

$F(-7) = 56$

$F(7) = 532 \longrightarrow$ Abs. max.

Ex) Find the abs. min. and/or abs. max of

$$f(x) = x^{-2} \ln x$$

on $[1, e]$.

(1) We found before the critical number of $f(x)$ is $\sqrt{e}$,

$$f(\sqrt{e}) = \sqrt{e}^{-2} \ln \sqrt{e}$$

$$= \frac{1}{\sqrt{e}^{2}} \ln e^{1/2}$$

$$= \frac{1}{e}\left(\frac{1}{2} \underbrace{\ln e}_{=1}\right) = \frac{1}{2e}$$

(2) $f(1) = 1^{-2} \ln 1 = 1 \cdot 0 = 0$

$f(e) = e^{-2} \ln e = e^{-2} = \frac{1}{e^2}$

(3) We get $f(\sqrt{e}) = \frac{1}{2e} \longrightarrow$ abs. max.

$f(1) = 0 \longrightarrow$ abs. min.

$f(e) = \frac{1}{e^2}$

Ex) Find the abs. max and/or abs. min of

$$f(x) = 2\cos x + \sin^2 x$$

on $[-\pi/2, \pi/2]$

(1) By a previous example, we know 0 is the only critical number in $[-\pi/2, \pi/2]$.

$$f(0) = 2\cos 0 + \sin^2 0$$
$$= 2$$

(2) $f(-\frac{\pi}{2}) = 2\cos\left(-\frac{\pi}{2}\right) + \sin^2\left(-\frac{\pi}{2}\right)$

Recall $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$ since $\cos$ is an even function. Also

$\sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$ since $\sin$ is an odd function.

$$f(-\pi/2) = 2\cdot 0 + (-1)^2 = 1$$

$$f(\pi/2) = 2\cdot 0 + 1^2 = 1$$

(3) $f(0) = 2 \rightarrow$ abs max.

$f(-\frac{\pi}{2}) = 1$
$f(\frac{\pi}{2}) = 1$ $\rightarrow$ abs min.

Exercises Find the abs. max and abs. min. of $f(x) = 12 + 4x - x^2$ on $[0, 5]$.

The Mean Value Theorem

Recall the Intermediate Value Theorem and that we use it to find a solution to the given equation. Now, we will provide other tools to decide the number of solutions to the given equation and to do more.

First, we will discuss Rolle's Theorem:

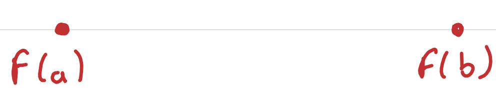
Rolle's Theorem Let f be a function such that

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)
3. $f(a)=f(b)$.

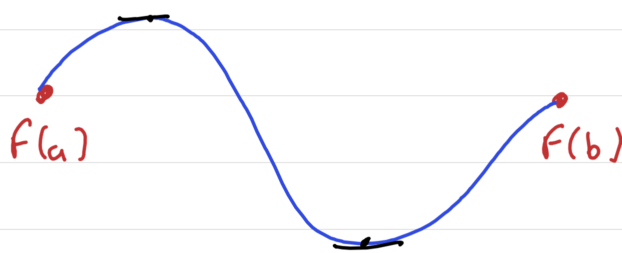
Then there is a number c in (a,b) such that $f'(c)=0$.

The proof is omitted but you can understand the idea from the assumptions. Suppose we are trying to sketch such a function,

Start with two points on the same horizontal line, they will be $f(a)$ and $f(b)$ which are the same.



Now connect two points in a continuous and differentiable way. Namely, no hole or no corner should appear.



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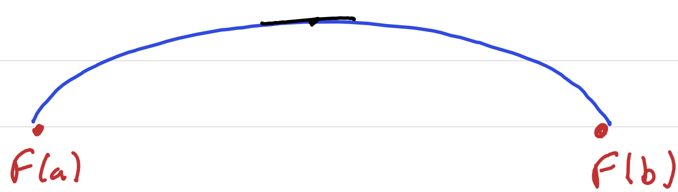

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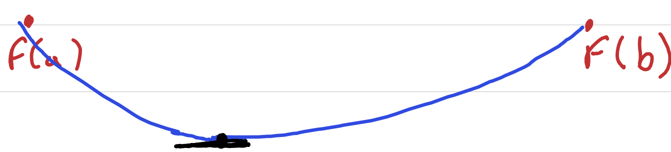


or

As you observed, in any case, we can find points c where $f'(c)=0$.

How we can use Rolle's Theorem?

Example: Prove that the equation

$$x^3+x-1=0$$

has exactly one real root.

Solution: First apply IVT to find a root. Let $f(x)=x^3+x-1$. Then

$$f(1)=1^3+1-1=1$$

$$f(0)=0+0-1=-1.$$

Since f is continuous, by IVT, there is a solution for the equation in $(0,1)$.

Second, we want to show that there is only one real root. We assume the contrary and try to achieve a contradiction.

Suppose there are two real numbers a, b such that $f(a) = a^3 + a^2 - 1 = 0$.

$$f(b) = b^3 + b^2 - 1 = 0.$$

Since f is continuous and differentiable, and $f(a) = 0 = f(b)$, by Rolle's Theorem, there should be a c in (a, b) such that $f'(c) = 0$. However

$$f'(x) = 3x^2 + 1 \geq 1 \quad \text{for all } x.$$

So $f'(c) = 0$ is impossible. Therefore, there cannot be two solutions a, b.

There is only one solution, so we are done ☺

<u>Remark</u> Rolle's Theorem is also important because it helps to prove the following important result:

(MVT)

Mean Value Theorem Let f be a function such that

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b).

Then there is a number c in (a,b) such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently,
$$f(b)-f(a) = f'(c)(b-a).$$

Examples Consider $f(x) = 2x^2 - 3x + 1$ on $[0,2]$.

Indeed, f is continous on $[0,2]$
differentiable on $(0,2)$.

By MVT, there is a number c in $(0,2)$ such that
$$f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{3-1}{2} = \frac{2}{2} = 1.$$

Since $f'(c) = 4c - 3$, $f'(c) = 1$ means $c = 1$.

As you see, c is indeed in $(0,2)$.

Example: Suppose $3 \leq f'(x) \leq 5$ for all x.
Show that $18 \leq f(8) - f(2) \leq 30$.

Solution: First of all since $f'(x)$ exists for all x, f is differentiable, and hence continuous.

Therefore, we can use MVT.

It means there is a c in $(2,8)$ such that

$$f'(c) = \frac{f(8)-f(2)}{8-2} = \frac{f(8)-f(2)}{6}.$$

By assumption, we know $3 \leq f'(c) \leq 5$.
So

$$3 \leq \frac{f(8)-f(2)}{6} \leq 5.$$

Multiply all sides with 6 and get

$$18 \leq f(8) - f(2) \leq 30.$$

We are done ☺

Remarks Our main focus for Rolle's Thm and MVT is that student should understand the hypothesis and the conclusion of these.

Examples Verify that the function

$$f(x) = x^3 - 2x^2 - 4x + 2$$

satisfies the three hypotheses of Rolle's Theorem on $[-2, 2]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

① $f(x)$ is continuous on $[-2,2]$ since it's a polynomial.

② $f(x)$ is differentiable on $(-2,2)$. Indeed,

$$f'(x) = 3x^2 - 4x - 4.$$

③ $f(-2) = (-2)^3 - 2(-2)^2 - 4(-2) + 2 = -8 - 8 + 8 + 2 = -6$

$f(2) = (2)^3 - 2(2)^2 - 4(2) + 2 = 8 - 8 - 8 + 2 = -6$

So $f(-2) = f(2)$

Conclusion

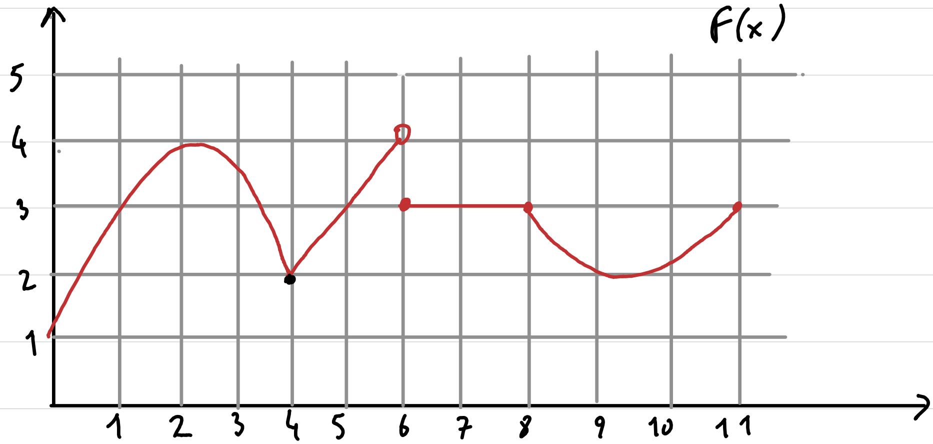
$\Longrightarrow$ There is a c in $(-2, 2)$ such that $f'(c) = 0$, namely

$$3c^2 - 4c - 4 = 0$$

$$(3c+2)(c-2) = 0$$

So $\boxed{c = -2/3 \text{ or } 2}$

Examples



On which intervals f satisfy the hypotheses of MVT? Select all apply

A [0,3] B [3,5] C [5,7] D [8,11]

Answer Recall that MVT needs

1. f is continuous on [a,b]
2. f is differentiable on (a,b).

Since f is not continuous at 6 and not differentiable at 4, f does not satisfy MVT on [3,5] and [5,7]. Correct options are [0,3] and [8,11]

Week 10 Objective

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y=f(x)$ by hand.

1. Find the domain of f, that is, the set of values of x for which $f(x)$ is defined.
2. The function can intersect with x axis, where $y=0$, or with y axis, where $x=0$. So find those points.
3. Find the asymptotes. Recall that

   $y=a$ is a horizontal asymptotes if

   $$\lim_{x\to\infty} f(x)=a \quad \text{or} \quad \lim_{x\to-\infty} f(x)=a,$$

   $x=b$ is vertical asymptotes if one of the following limits is ∞ or $-\infty$:

   $$\lim_{x\to b} f(x), \quad \lim_{x\to b^+} f(x), \quad \lim_{x\to b^-} f(x).$$

4. Find the intervals of increase or decrease.

   $f'(x)>0$ on (a,b) $\Rightarrow$ f is increasing on (a,b)

   $f'(x)<0$ on (a,b) $\Rightarrow$ f is decreasing on (a,b)

(5) Find the local extremum points.

Find the critical numbers of f and apply First Derivative Test (or Second Derivative Test) to determine whether there is a local minimum or maximum.

(6) Find the points of inflections and the intervals of concavity.

Apply Concavity Test.

(7) Collect all previous data and draw the graph.

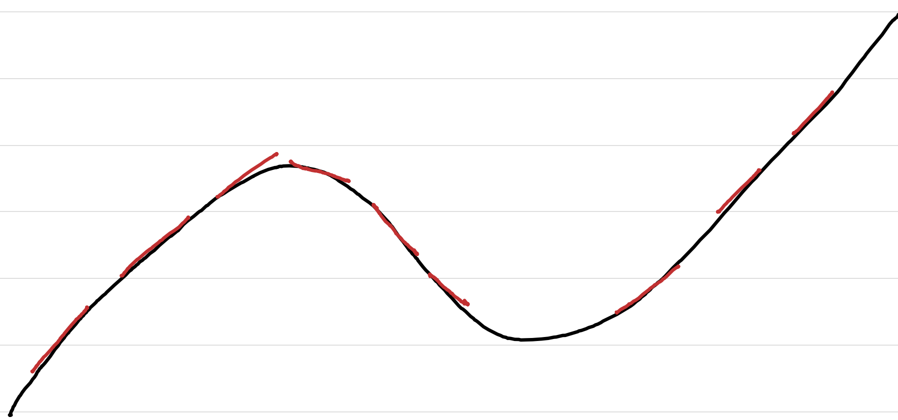
We already covered the first three steps. The upcoming lectures will cover steps 4, 5, 6, and then we expect to solve given graph sketch questions. We also provide an example question from the previous final exams.

What does f' say about f?

Fact ① If $f'(x) > 0$ on an interval, then f is increasing on that interval.

Fact ② If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

You can observe it from the slopes of the tangent line in both cases.



As you might observe when the sign of $f'(x)$ changes, there is a local extremum point, since it behaves either like $\nearrow \searrow$ or $\searrow \nearrow$

Fact ③ $\boxed{\text{First Derivative Test}}$

Suppose c is a critical number of a continuous function f,

a) If f' changes from $+$ to $-$ at c, then f has a local maximum at c.

b) If f' changes from $-$ to $+$ at c, then f has a local minimum at c.

c) If there is no change at c, then there is no local extremum at c.

Examples $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. Then

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$,

So the critical points are $0, 2, -1$. Using the following chart, we can apply the test.

interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x<-1$	$-$	$-$	$-$	$-$	decreasing
				change	
$-1<x<0$	$-$	$-$	$+$	$+$	increasing
				change	
$0<x<2$	$+$	$-$	$+$	$-$	decreasing
				change	
$2<x$	$+$	$+$	$+$	$+$	increasing

At $x = -1, 2$ local minimum, at $= 0$ local maximum.

Example: Find the intervals of increase/decrease and local min/max of

$$f(x) = \frac{x}{x^2+1}.$$

Solution:
$$f'(x) = \frac{(x)'(x^2+1) - (x)(x^2+1)'}{(x^2+1)^2}$$
$$= \frac{x^2+1 - x(2x)}{(x^2+1)^2}$$
$$= \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

Critical numbers are 1 and -1 since $f'(1) = 0 = f'(-1)$.

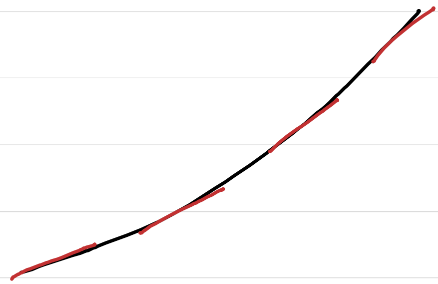
intervals	$1-x$	$1+x$	$(x^2+1)^2$	$f'(x)$	f
$x<-1$	+	−	+	−	decreasing
				change	
$-1<x<1$	+	+	+	+	increasing
				change	
$1<x$	−	+	+	−	decreasing

f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.
f is increasing on $(-1, 1)$.
At $x = -1$ f has local minimum.
At $x = 1$ f has local maximum.

What does f'' say about f?

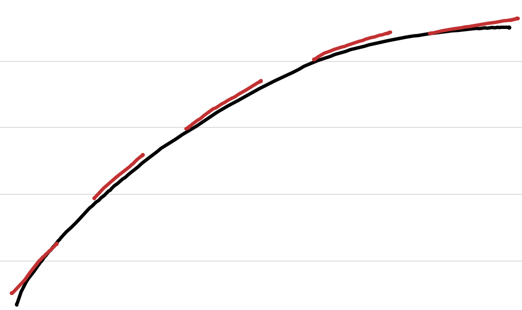
Definition: If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I.

Ex.



If the graph of f lies below all of its tangents on I, then it is called concave downward on I.

Ex.



Definition: A point P on a curve $y=f(x)$ is called inflection point if f is continuous and the curve changes from concave upward to downward or from concave downward to upward.

Fact ④ $\boxed{\text{Concavity Test}}$

a) If $f''(x)>0$ on an interval, then f is concave upward on that interval.

b) If $f''(x)<0$ on an interval, then f is concave downward on that interval.

Example: $f(x)=x^4-4x^3$, then $f''(x)=12x(x-2)$, and $f''(x)=0$ for $x=0, 2$. Using the chart below, we can apply the test.

interval	$12x$	$x-2$	$f''(x)$	f
$x<0$	$-$	$-$	$+$	concave up
			change	
$0<x<2$	$+$	$-$	$-$	concave down
			change	
$2<x$	$+$	$+$	$+$	concave up

At $x=0, 2$ f has inflection points.

Example: Find the intervals of concavity and inflection points of

$$f(x) = x^4 - 2x^2 + 3.$$

Solution: $f'(x) = 4x^3 - 4x$

$$f''(x) = 12x^2 - 4 = 12\left(x^2 - \tfrac{1}{3}\right)$$

$$= 12\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)$$

Intervals	$x - 1/\sqrt{3}$	$x + 1/\sqrt{3}$	$f''(x)$	f
$x < -\frac{1}{\sqrt{3}}$	$-$	$-$	$+$	concave up
			change	
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$-$	$+$	$-$	concave down
			change	
$\frac{1}{\sqrt{3}} < x$	$+$	$+$	$+$	concave up.

f is concave upward on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

f is concave downward on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

f has inflection points at $x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

Fact 5 $\boxed{\text{Second derivative test}}$

Suppose f'' is continuous near c.

(a) If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.

(b) If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.

Remark) To find local extremums, you can use first or second derivative test. It totally depends your preference.

Ex) Find the local max./min. of

$$f(x)=1+3x^2-2x^3$$

using the second derivative test.

Solution) First, find c such that $f'(c)=0$.

$f'(x)=6x-6x^2=6x(1-x)$.

So $f'(0)=0$ and $f'(1)=0$.

To use second derivative test, find f''.

$f''(x)=6-12x$.

Now, $f''(0) = 6 > 0$
$f''(1) = -6 < 0$

So there is a local minimum at $x=0$, and there is a local maximum at $x=1$.

The local min. value is $f(0)=1$.
The local max. value is $f(1)=2$.

Now, we covered all necessary info to sketch a graph. Most parts relies on derivative computations, but finding asymptotes require to find limits.

Before giving examples about graph sketchs, we'll cover a nice technique to deal with asymptotes easily. This technique will be also useful for many limit calculations.

Indeterminate Forms and L'Hospital's Rule

Recall the examples like

$$\lim_{x\to 0} \frac{\sin x}{x} \quad , \quad \lim_{x\to 0^+} x\ln x \quad , \quad \lim_{x\to\infty} x - e^x .$$

In the first case, since $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} x = 0$ we have something like

$$\frac{0}{0} .$$

In the second case, since $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, we have something like

$$0 \cdot (-\infty)$$

In the third case, since $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we have something like.

$$\infty - \infty$$

All these expressions are called <u>indeterminate forms</u>. There are more forms like those:

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, 0\cdot\infty, \infty-\infty,$$

$$0^0, \infty^0, 1^\infty.$$

We mean here suppose we have functions $f(x)$ and $g(x)$ and we may have the following cases:

$\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty$, so

- $\lim_{x\to a} f(x)\,g(x)$ is like $0\cdot\infty$
- $\lim_{x\to a} f(x)^{g(x)}$ is like 0^∞
- $\lim_{x\to a} \frac{f(x)}{g(x)}$ is like $\frac{0}{\infty}$
- and so on ...

We can handle these cases using the method called L'Hospital's Rule:

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a. Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

OR

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty,$$

Namely we have indeterminate forms like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists (or is $\pm\infty$).

Examples

(1) $\lim_{x\to 1} \frac{\ln x}{x-1}$.

We have $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} x-1 = 0$.

Using L'Hospital, we get

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{(\ln x)'}{(x-1)'}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$$

(2) $\lim_{x\to\infty} \frac{e^x}{x^2}$.

We have $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$.

Using L'Hospital, we get

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

apply L'Hospital again

③ $\lim_{t \to 1} \frac{t^8 - 1}{t^5 - 1}$,

$\lim_{t \to 1} t^8 - 1 = 0$, $\lim_{t \to 1} t^5 - 1 = 0$. So we have indeterminate form of $\frac{0}{0}$.

Using L'Hospital, we get

$$\lim_{t \to 1} \frac{t^8 - 1}{t^5 - 1} = \lim_{t \to 1} \frac{8t^7}{5t^4} = \lim_{t \to 1} \frac{8}{5} t^3 = \frac{8}{5}.$$

Remark: If we have indeterminate forms like $0 \cdot \infty$ or $\infty - \infty$, we can first try to make them like $\frac{0}{0}$, $\frac{\infty}{\infty}$, then apply L'Hospital's Rule.

Example: $\lim_{x \to \infty} x \sin\left(\frac{\pi}{x}\right)$.

Since $\lim_{x \to \infty} x = \infty$ and $\lim_{x \to \infty} \sin\left(\frac{\pi}{x}\right) = 0$,

we have indeterminate form of $\infty \cdot 0$.
However, we can do more.

$$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}.$$

since $x = \frac{1}{\frac{1}{x}}$

Now we have indeterminate form of $\frac{0}{0}$.
We can use L'Hospital :

$$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$$

note that $\frac{\pi}{x} = \pi x^{-1}$, so derivative is $-\pi x^{-2}$.

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \pi \cos\left(\frac{\pi}{x}\right) = \pi \underbrace{\lim_{x\to\infty} \cos\left(\frac{\pi}{x}\right)}_{=1} = \pi$$

! Note that we achieved

$$\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = 1.$$

So we also achieved a horizontal asymptote for $f(x) = x \sin\left(\frac{\pi}{x}\right)$ which is $y = 1$.

<u>Example:</u> Calculate $\lim\limits_{x\to\infty} e^x - x$.

We have indetermine form of $\infty - \infty$. However, we can do more.

$$\lim_{x\to\infty} e^x - x = \lim_{x\to\infty} x\left(\frac{e^x}{x} - 1\right).$$

Now, $\lim\limits_{x\to\infty} x = 0$ and

$$\lim_{x\to\infty} \frac{e^x}{x} - 1 = \infty \quad \text{since} \quad \lim_{x\to\infty} \frac{e^x}{x} \overset{\text{L'Hospital.}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

So $\lim\limits_{x\to\infty} e^x - x = \lim\limits_{x\to\infty} x\left(\frac{e^x}{x} - 1\right) = \infty \cdot \infty = \infty$.

GRAPH SKETCH EXAMPLES.

Recall the guideline:

(1) Find the domain

(2) Find x-intercepts and y-intercepts

(3) Find the asymptotes

(4) Find intervals of increase /decrease

(5) Find local extremums

(6) Find concavity intervals and points of inflection.

(7) Finish sketching

Examples Sketch the curve $y = \frac{x}{x-1}$.

(1) The domain is $\mathbb{R} \setminus \{1\}$ since $\frac{x}{x-1}$ is not defined at 1.

(2) If $x=0$, then $y=0$. So the curve passes through the point $(0,0)$

(3) For vertical asymptote, check

$\lim_{x \to 1} \frac{x}{x-1} = \infty$, so $x=1$ is V.A.

For horizontal asymptote, check

$$\lim_{x \to \infty} \frac{x}{x-1} \underset{\text{L'Hospital}}{=} \lim_{x \to \infty} \frac{1}{1} = 1$$

Similarly $\lim_{x \to -\infty} \frac{x}{x-1} = 1$, So $y = 1$ is a H. A.

(4) Check $F'(x) = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$ Quotient rule

Only critical number is 1, but it is not in the domain. But observe that

$$F'(x) = -\frac{1}{(x-1)^2} < 0 \quad \text{For all } x.$$

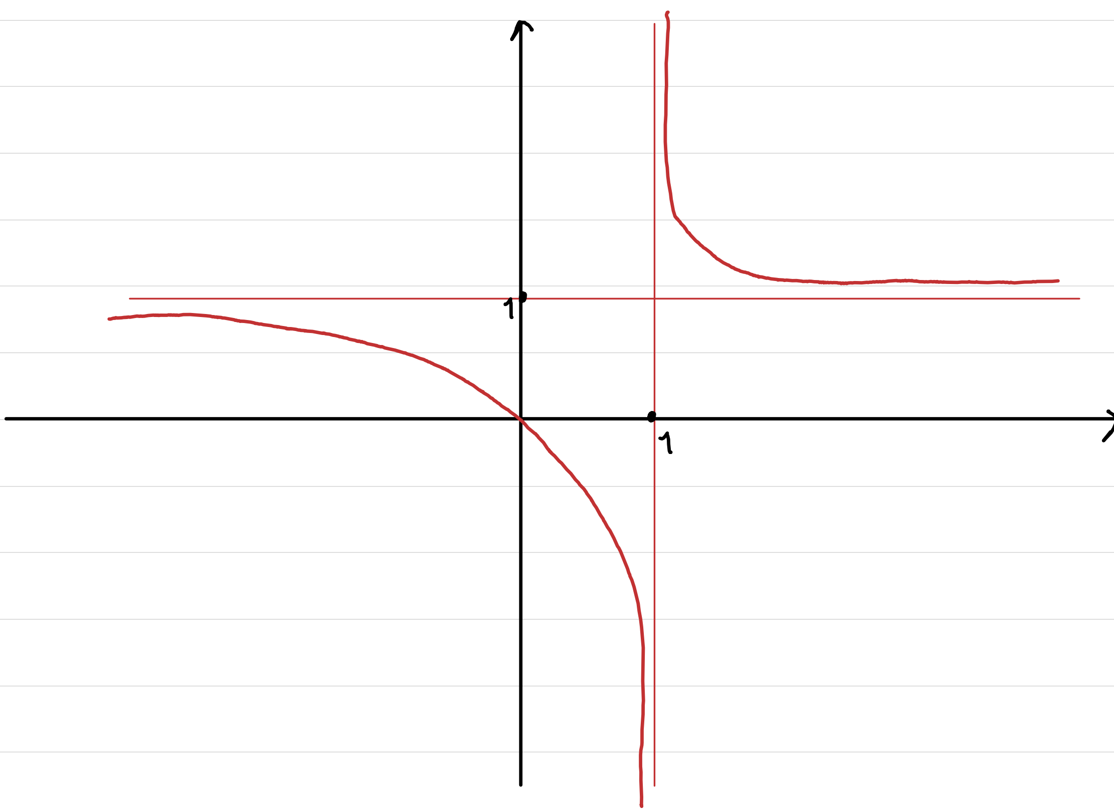
So F is decreasing everywhere.

(5) Since there is no critical numbers or boundary points, we don't need to check local min. /max.

(6) $f''(x) = \dfrac{2}{(x-1)^3}$, Again $x=1$ is critical.

interval	$f''(x)$	f
$x<1$	$-$	concave down
$1<x$	$+$	concave up

There is no point of inflection at $x=1$, since it's not in the domain.



Examples Sketch the curve $y = x(x-4)^3$.

① Domain is $\mathbb{R}$.

② If $x=0, y=0$.
If $y=0$, $x=0$ or 4. So the curve is passing through $(0,0)$ and $(4,0)$.

③ No vertical asymptote since the curve is defined everywhere.

$$\lim_{x\to\infty} x(x-4)^3 = \infty \quad \text{and} \quad \lim_{x\to-\infty} x(x-4)^3 = \infty$$

So no horizontal asymptote.

④ $f'(x) = (x-4)^3 + 3x(x-4)^2 = (x-4)^2(x-4+3x)$
$= (x-4)^2(4x-4)$

Critical numbers are $4, 1$.

interval	$(x-4)^2$	$4x-4$	$f'(x)$	f
$x<1$	+	−	−	decreasing
			change	
$1<x<4$	+	+	+	increasing
$4<x$	+	+	+	increasing

So the curve is increasing on $(1, \infty)$
decreasing on $(-\infty, 1)$.

(5) By previous part, there is a local min. at $x=1$. The value is

$$f(1) = 1(1-4)^3 = (-3)^3 = -27.$$

(6) $f''(x) = 3(x-4)^2 + 3(x-4)^2 + 6x(x-4)$

$$= 3(x-4)(x-4+x-4+2x)$$
$$= 3(x-4)(4x-8)$$

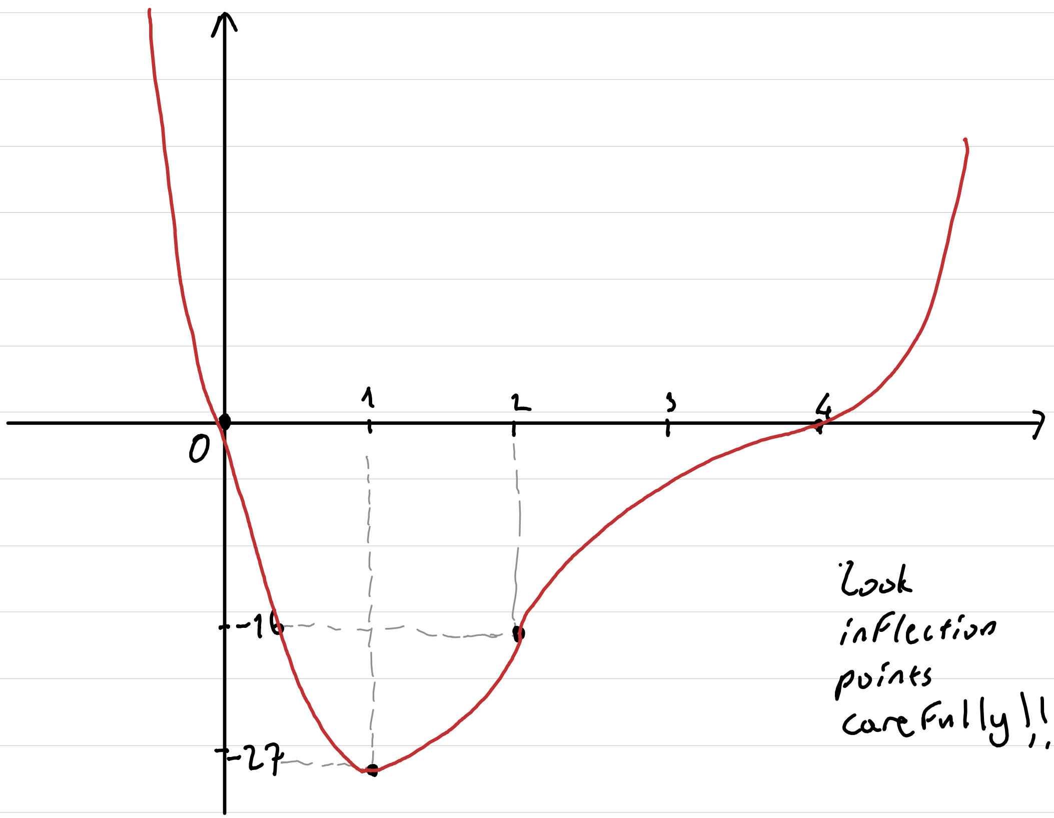
$f''(x)=0$ for $x=4$ or 2.

interval	$3(x-4)$	$4x-8$	$f''(x)$	f
$x<2$	$-$	$-$	$+$	concave up
			change	
$2<x<4$	$-$	$+$	$-$	concave down
			change	
$4<x$	$+$	$+$	$+$	concave up

At $x=2, 4$, there are points of inflections.
The values are are

$$f(2) = 2(2-4)^3 = -16$$
$$f(4) = 2(4-4)^3 = 0$$

(7) Start with specifying the important points, then connect them using (3) and (6) steps.



Optimization Problems

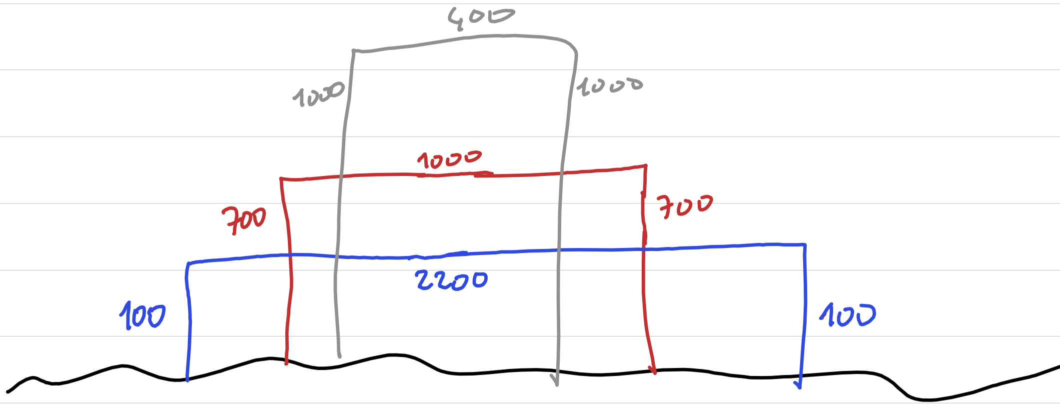
Finding extreme values have practical applications in many areas. Minimizing the cost, maximizing the profit, finding the shortest path, minimizing the transportation time, maximizing the speed, etc. We want to achieve these.

For given problem, we'll formulate it using functions and optimization will be given by extreme values of the function.

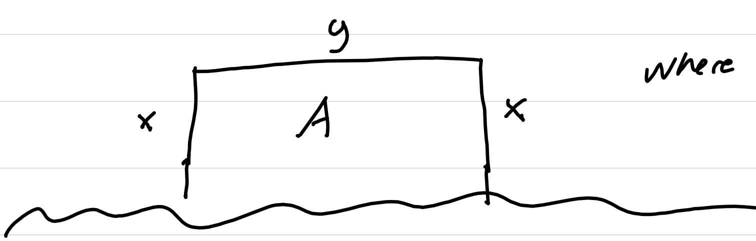
Before mentioning the related tool, let's analyze an example.

Ex) A farmer has 2400 ft^2 of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution: Let's depict the situation.



There are different options, but we want to achieve possible largest area. So the general case is



where $2x+y=2400$

$A=xy$.

So we want to maximize A. Using the constraint $2x+y=2400$, we can do $y=2400-2x$, and write

$$A=x(2400-2x)=2400x-2x^2. \quad (\otimes)$$

Now, we reduce the problem to find maximum of $\otimes$.

Namely, we want to maximize the function

$$A(x) = 2400x - 2x^2,$$

Note that the constraint $2x + y = 2400$ says another thing. x and y can be at least one. If x increases then y decreases, and vice versa. So if $y = 0$, then $x = 1200$. This means x can be at most 1200. So the domain of $A(x)$ is $[0, 1200]$.

Let's apply the general method:

- First, find the critical numbers

$$A'(x) = 2400 - 4x = 0$$

$$2400 = 4x$$

$$600 = x$$

- Evaluate

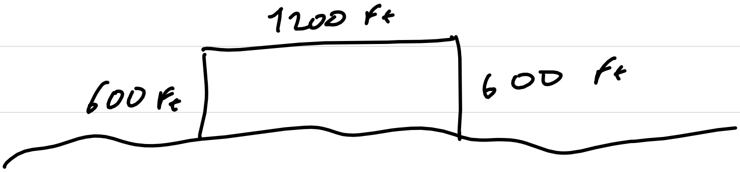
$$A(0) = 0$$

$$A(1200) = 0$$

$$A(600) = 720,000 \quad \text{maximum.}$$

So $x = 600$ and then $y = 2400 - 2x = 1200$.

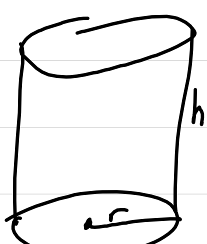
So the desired dimension is



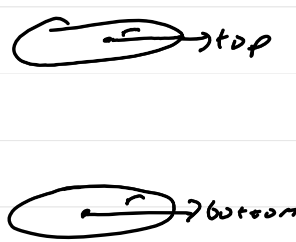
This problem is solved using a function on bounded domain "[0, 1200]", so we check those points and critical numbers to get the absolute maximum. But we may have different situation.

Example: A cylindrical can is to be made to hold $1000\ cm^3$ of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution:



Total area:



So the total area is

$$\underbrace{\pi r^2}_{\text{top}} + \underbrace{\pi r^2}_{\text{bottom}} + \underbrace{2\pi r(h)}_{\text{sides}} = 2\pi r^2 + 2\pi r h \quad \emptyset$$

What we are given: the volume is $1000\,cm^3$.
So

$$V = \pi r^2 h = 1000.$$

We can say $h = \dfrac{1000}{\pi r^2}$.

So if we put this into $\emptyset$, the area we want to minimize is

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}$$

Now, r can be anything greater than 0.
So how we find absolute minimum of $A(r)$?

Again use critical numbers.

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}$$

If $A'(r) = 0$, then $4\pi r^3 - 2000 = 0$

$$4\pi r^3 = 2000$$

$$\pi r^3 = 500$$

$$r = \sqrt[3]{\frac{500}{\pi}},$$

interval	$A'(r)$	$A(r)$
$0 < r < \sqrt[3]{\frac{500}{\pi}}$	$-$	decreasing
$\sqrt[3]{\frac{500}{\pi}} < r$	$+$	increasing

So, yes at $r = \sqrt[3]{\frac{500}{\pi}}$ there is a minimum, but since there is no other change, we can take it as an absolute minimum.

So $r = \sqrt[3]{\frac{500}{\pi}}$, then

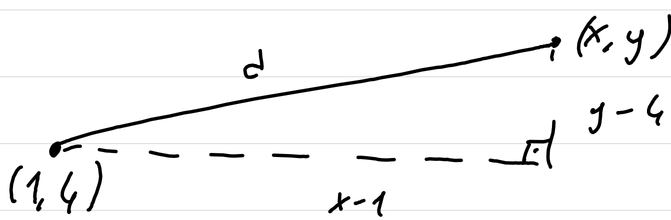
$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{2/3}} = 2\left(\frac{500}{\pi}\right)\left(\frac{500}{\pi}\right)^{-2/3} = 2\left(\frac{500}{\pi}\right)^{1/3} = 2r.$$

So the desired dimension to minimize production cost is

$$r = \sqrt[3]{\frac{500}{\pi}} \text{ cm} \quad \text{and} \quad h = 2r.$$

Example: Find the point on the curve $y^2 = 2x$ that is closest to the point $(1, 4)$.

Solution: First, we need the distance function: We have two points $(1,4)$ and (x,y) but the latter is on the curve.



So the distance $d = \sqrt{(x-1)^2 + (y-4)^2}$.

However, we know $y^2 = 2x$, namely, $y^2/2 = x$.

So we can write

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}.$$

You can use this function directly, but minimizing it needs derivative involving derivative of square root. Let's not bother with it and use

I gave this name for the function

$$d^2 = f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

because minimizing d is the same as minimizing d^2.

$$f'(y) = 2\left(\frac{y^2}{2} - 1\right) y + 2(y-4)$$

$$= y^3 - 2y + 2y - 8 = y^3 - 8$$

If $y^3 - 8 = 0$, then $y^3 = 8$, so $y = 2$.

	$f'(y)$	$f(y)$
$y<2$	$-$	decreasing
$y<2$	$+$	increasing

So as in the previous example, we say at $y=2$, we have absolute minimum.

So if $y=2$, then using the curve $y^2 = 2x$, we get $x=2$. So the closest point on the curve to $(1,4)$ is $(2,2)$.

Examples A company sells x units of a product at a price $p = 200 - 2x$ dollars. The cost of producing x units is $C(x) = 40x + 300$. Find the quantity x that maximizes the profit.

Solution What is the profit?

It is

Revenue − Cost.

What is the revenue?

It is price × number of product sold.

So profit function $P(x)$ is given by

$$P(x) = (200-2x)\,x - (40x+300)$$

$$P(x) = 200x - 2x^2 - 40x - 300$$

$$\boxed{P(x) = -2x^2 + 160x - 300}$$

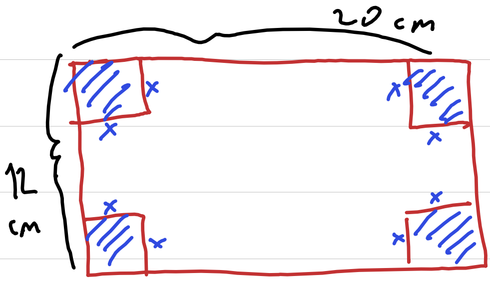
We want to find its maximum value.

$P'(x) = -4x + 160$. If $P'(x) = 0$, then $x = 40$.

	$P'(x)$	$P(x)$
$x < 40$	$+$	increases
$40 < x$	$-$	decreases.

So at $x = 40$, there is an abs. max.
To get maximum profit, the company should sell 40 products.

Exercise: A rectangular piece of cardboard with dimension 12 cm by 20 cm is to be made into an open-top box by cutting squares of equal size from each corner and folding up the sides. Find the size of the square that should be cut to maximize the volume of the box.



TASK:
Find the volume of the box we obtain and find its maximum value to find x.

Antiderivatives

Suppose you already have a change of rate info about some entity "x" and you want to know exactly about x. For example, you know the velocity, but want to know position. You know the change of rate for a population, but want to know the population.
In summary, you have a function $f(x)$, and want $F(x)$ where $F'(x) = f(x)$.

Definition: A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Examples:

- $F(x) = \dfrac{x^2}{2}$ is antiderivative of $f(x) = x$ since $F'(x) = x = f(x)$.

- $F(x) = -\cos x$ is antiderivative of $f(x) = \sin x$ since $F'(x) = -(-\sin x) = \sin x = f(x)$.

Remark Note that, we also have

$$G(x) = \frac{x^2}{2} + 5 \text{ is antiderivative of } f(x) = x$$

since $G'(x) = x = f(x)$. This is true for all functions of the form $\frac{x^2}{2} + C$ where C is a number.

Therefore, we have a general result:

Theorem If F is an antiderivative of f on I, then the <u>most general</u> antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

How we find antiderivatives? We basically reverse back the derivative. Namely, if you have a function $f(x)$, try to get a function $F(x)$ such that after applying derivative on F, you get f. For example, if you have $f(x) = x^2$, you can guess x^3 can be the previous function before taking the derivative.

But we need $\frac{x^3}{3}$ since the derivative

of x^3 is $3x^2$, but we expect x^2. Adding the general constant C to it, we get

$$F(x) = \frac{x^3}{3} + C$$ is the

antiderivative of $f(x) = x^2$.

Let's give all antiderivatives for the essential functions.

① $f(x) = x^n$, $n \neq -1$. Applying power rule reversely, we can guess, the antiderivative involves x^{n+1}. But we need $\frac{x^{n+1}}{n+1}$ to get rid of coefficient $n+1$ after taking the derivative. To generalize, we need constant C. So the antiderivative of x^n for $n \neq -1$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C.$$

(2) $f(x) = \frac{1}{x}$.

Which function has the derivative $\frac{1}{x}$?
Recall that this is $\ln x$. But we're searching for the most general case.
We also have

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Why this is more general? Because its domain is $\mathbb{R}\setminus\{0\}$ which is the same as the domain of $\frac{1}{x}$. Adding the constant C, we get

$F(x) = \ln|x| + C$ is the antiderivative of $\frac{1}{x}$.

(3) $F(x)=e^x$. As you might guess, the derivative of e^x is itself, so adding the constant, we get

$$F(x)=e^x+C$$

is the antiderivative of e^x.

(4) $f(x)=b^x$, The derivative of it is $b^x \ln b$. So if we get rid of $\ln b$, then we can get b^x, It means we need $\frac{b^x}{\ln b}$, Adding the constant, we get

$$F(x)=\frac{b^x}{\ln b}+C$$

is the antiderivative of b^x.

(5) $f(x) = \cos x$. Since the derivative of $\sin x$ is $\cos x$, we directly get, adding the constant, that

$$F(x) = \sin x + C$$

is the antiderivative of $f(x) = \cos x$.

Similary

$F(x) = -\cos x + C$ is antiderivative of $f(x) = \sin x$.

$F(x) = \tan x + C$ is antiderivative of $f(x) = \sec^2 x$.

$F(x) = \sec x + C$ is antiderivative of $f(x) = \sec x \tan x$

etc...

(6) Like we did in the derivatives, we have some rules related to antiderivatives.

(a) If $F(x)$ is antiderivative of $f(x)$, then $cF(x)$ is antiderivative of $cf(x)$.

(b) If $F(x)$ is antiderivative of $f(x)$ and $G(x)$ is antiderivative of $g(x)$, then $F(x)+G(x)$ is antiderivative of $f(x)+g(x)$.

So we can solve the examples as below:

Ex) Find the antiderivative of

$$x^2 - 3x + 2.$$

Solution) We need an $F(x)$ such that

$$F'(x) = x^2 - 3x + 2.$$

Applying the rule about powers, we get

$$F(x) = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x + C$$

don't forget adding this constant.

You can easily verify

$$F'(x) = \frac{3x^2}{3} - \frac{6x}{2} + 2 = x^2 - 3x + 2.$$

Example: Find the function f if we have
$$f'(x) = \frac{1}{5} - \frac{2}{x}.$$

Applying the rules we did before, we get
$$f(x) = \frac{1}{5}x - 2\ln|x| + C.$$

Remark: You may ask "Why we don't add the constant C twice; one for $\frac{1}{5}$ and one for $\frac{2}{x}$?" You can add, but since these are two arbitrary constants C_1 and C_2, their sum $C_1 + C_2$ would be again arbitrary. Thus, it's enough to add a general constant C.

Example: Find the function $g(x)$ where
$$g''(x) = 12x^2 + 6x - 4.$$

Solution: In this example, we know the second derivative. So in order to reach the original function, we have two steps:

① If $g''(x) = 12x^2 + 6x - 4$,

then $g'(x) = 12\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) - 4x + C$

$g'(x) = 4x^3 + 3x^2 - 4x + C$.

② If $g'(x) = 4x^3 + 3x^2 - 4x + C$,

then $g(x) = 4\left(\frac{x^4}{4}\right) + 3\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + Cx + D$

$= x^4 + x^3 - 2x^2 + Cx + D$.

Remark: If we have more info about given function, we can find the exact function.

Ex: Find $g(x)$ if $g''(x) = 12x^2 + 6x - 4$ and. $g(0) = 4$, $g(1) = 1$.

Solution: We already found

$g(x) = x^4 + x^3 - 2x^2 + Cx + D$.

Now, we can find C and D using given values.

$$4 = g(0) = 0^4 + 0^3 - 2\cdot 0^2 + C\cdot 0 + D = D.$$

So $D=4$. Then use $g(1)=1$ to find C.

$$1 = g(1) = 1^4 + 1^3 - 2\cdot 1^2 + C\cdot 1 + 4 = C+4.$$

So $C=-3$. This means, we have

$$g(x) = x^4 + x^3 - 2x^2 - 3x + 4.$$

Examples Find f where

$$f''(x) = \sin x + \cos x,$$
$$f(0) = 3,$$
$$f'(0) = 4.$$

Solutions Let's find general solution step by step.

If $f''(x) = \sin x + \cos x$, then

$$f'(x) = -\cos x + \sin x + C.$$

But we have $f'(0) = 4$. So

$$4 = f'(0) = -\cos 0 + \sin 0 + C = -1 + C.$$

It means $C = 5$.

If $f'(x) = -\cos x + \sin x + 5$, then

$f(x) = -\sin x - \cos x + 5x + D$.

But we know $f(0) = 3$, so

$3 = f(0) = -\sin 0 - \cos 0 + 5 \cdot 0 + D = -1 + D$,

namely $D = 4$. Therefore

$$f(x) = -\sin x - \cos x + 5x + 4.$$

Let's verify:

$f'(x) = -\cos x - (-\sin x) + 5$

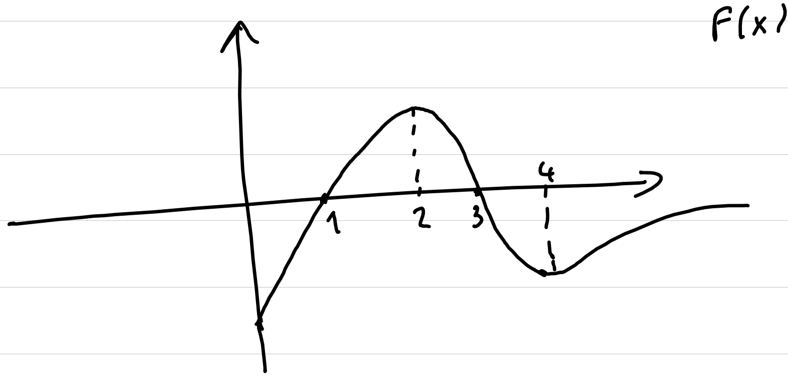
$= -\cos x + \sin x + 5$

$f''(x) = -(-\sin x) + \cos x$

$= \sin x + \cos x$ ✓

We can also make a rough sketch of an antiderivative of a given function with graph.

Example

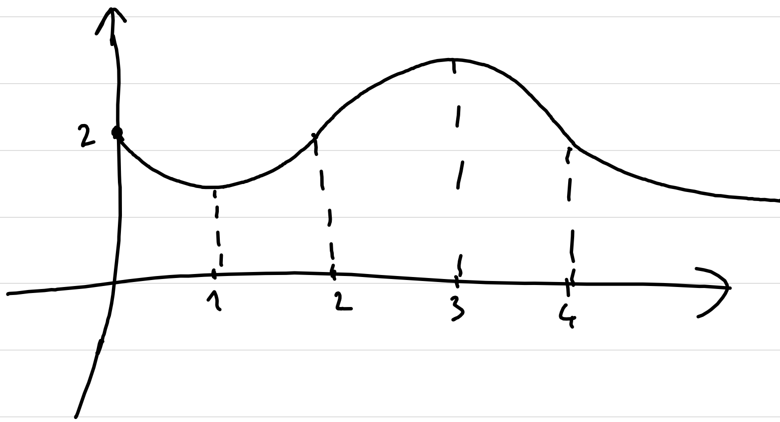


The graph of $f(x)$ is given. We want to sketch of an antiderivative $F(x)$, given that $F(0)=2$. Don't forget! $F'(x)=f(x)$. So we're looking at the graph of the derivative of F.

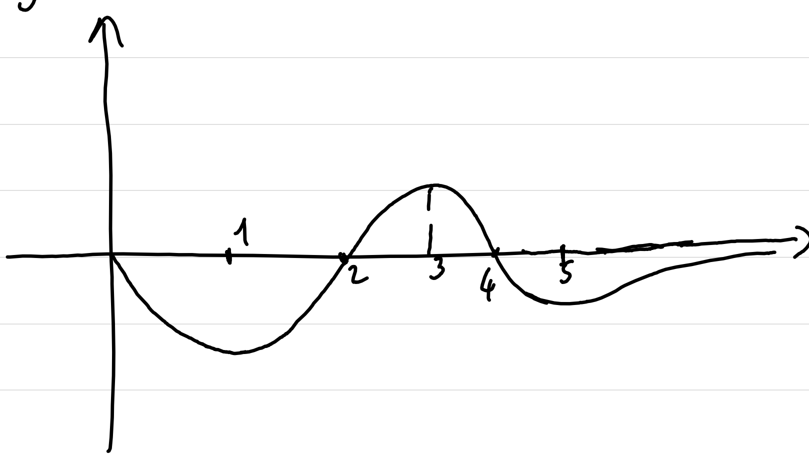
We can observe

- $F'(x)<0$ on $(0,1)$ and $(3,\infty)$. So F is decreasing on those intervals.
- $F'(x)>0$ on $(1,3)$, so F is increasing.
- $F'(x)$ changes from $-$ to $+$ at $x=1$, so F has a local min. at $x=1$.
- $F'(x)$ changes from $+$ to $-$ at $x=3$, so F has a local max. at $x=3$.
- Also, since $F'(x)$ increases on $(0,2)$ and $(4,\infty)$, and decreases on $(2,4)$, we

can also obtain the concavity and inflection points. Accordingly, we get

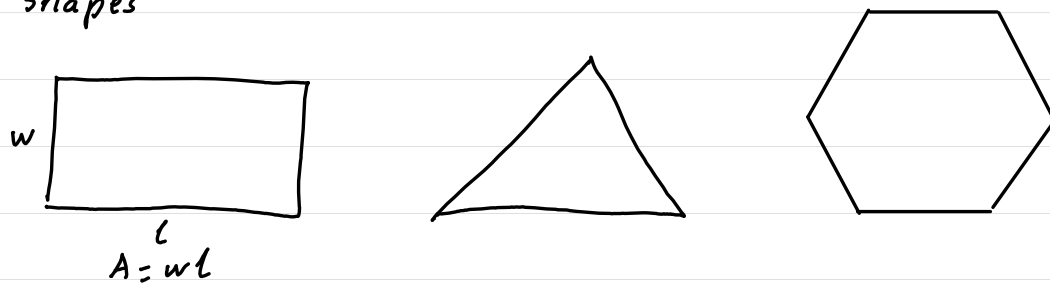


Exercise The graph of a function is shown in the figure below. Make a rough sketch of an antiderivative F, given that $F(0)=1$.

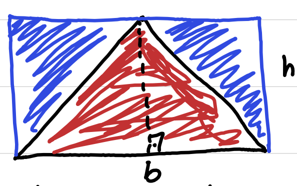


Area Problem

When we ask "Find the area" for the following shapes

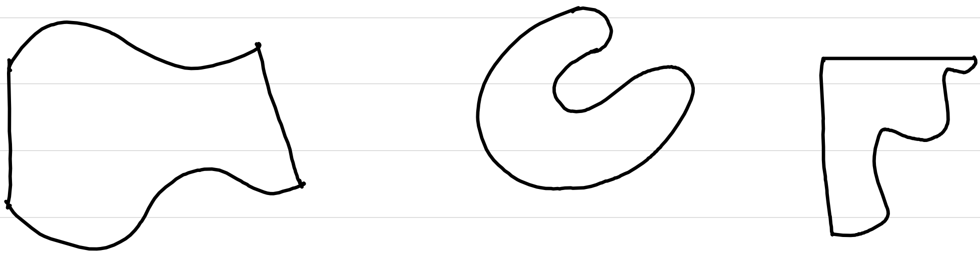


you can use basic formulas for them and get the answer easily. Actually, it is enough to know the area of rectangle, and you can find the areas of triangles, hexagons, etc. using it. For example, in the case of a triangle



whe have red triangle lives in a rectangle and the blue triangle is the same as red triangle. That's why the area is $\frac{hb}{2}$.

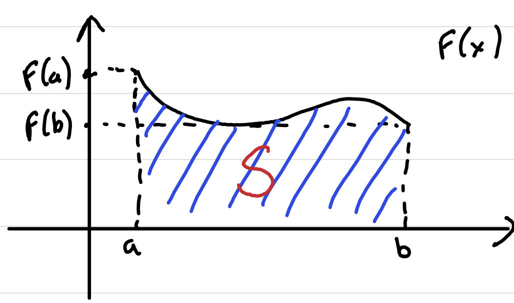
But what about the shapes with curved sides?



etc. Is there any formula for such shapes? If there is, for each different shape, do we have different formulas? How we can find these? We'll solve those question with a simple answer.

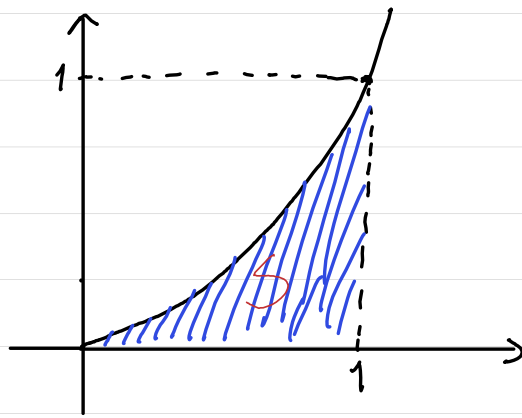
The area under the curve

Consider the following graph:

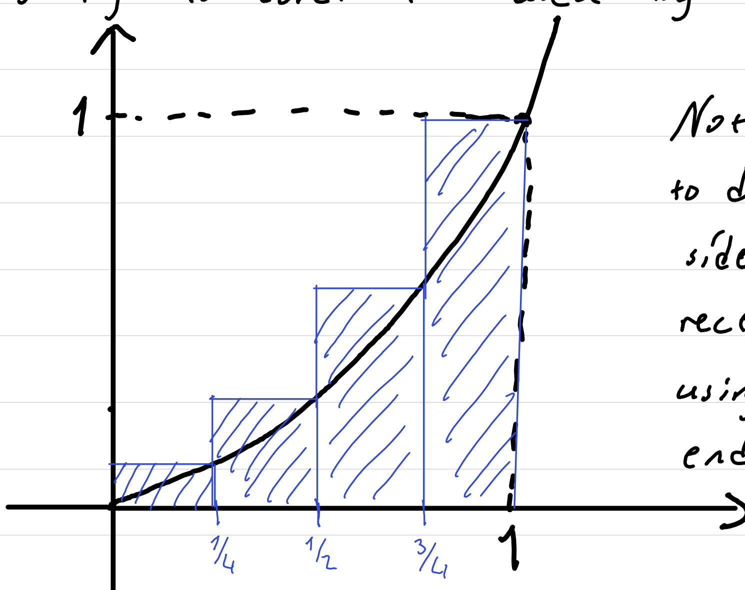


We want to find the area S which is the area under the curve $y = f(x)$. How?

Let's discuss an example first: consider the graph of x^2 on $[0,1]$

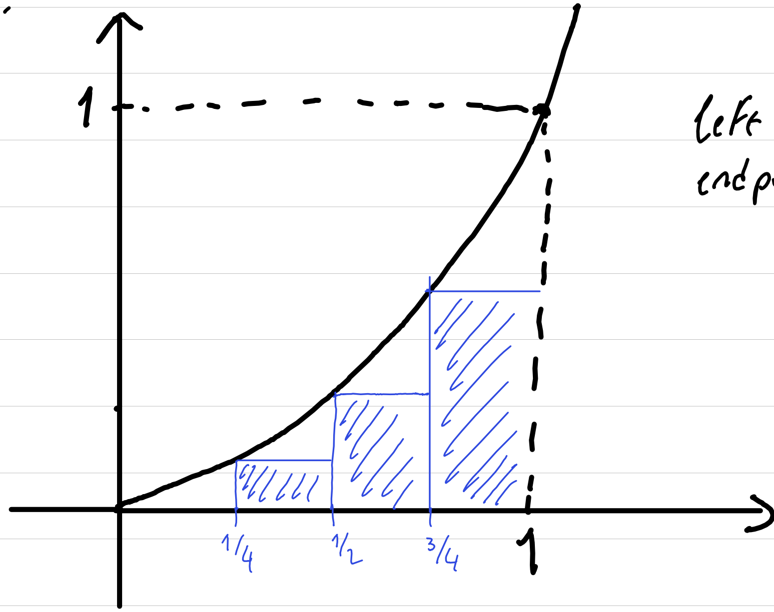


We'll find the area of blue region S. Like in the case of triangle, we can use rectangles. Let's try to cover the area by rectangles

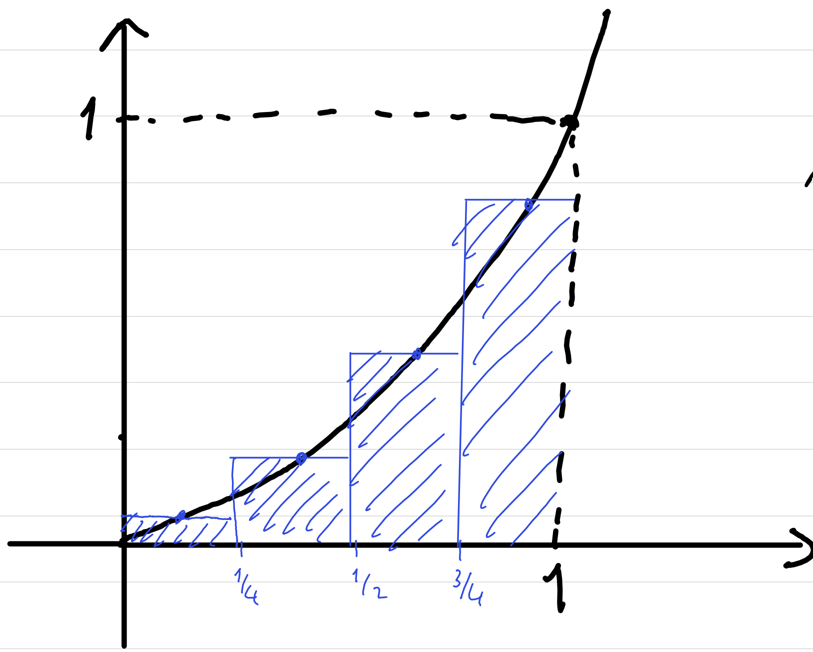


Note that to decide the sides of a rectangle, we're using right end points.

But this is not the only way to get rectangles. We can use left endpoints or midpoints:



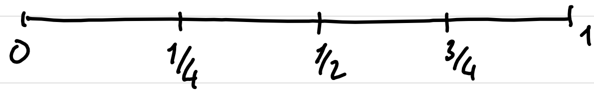
left endpoints



midpoints

Here, the idea how we get the rectangles is as follows:

- We divide $[0,1]$ into 4 equal pieces



- In right endpoint case:

  we use the right endpoints of each part, namely,

  $[0, \frac{1}{4}]$ $\quad [\frac{1}{4}, \frac{1}{2}]$ $\quad [\frac{1}{2}, \frac{3}{4}]$ $\quad [\frac{3}{4}, 1]$

  and the length of the rectangles are the outputs of these endpoints.

- In left endpoint case:

  we use the left enpoints of each part,

  namely $[0, \frac{1}{4}]$ $\quad [\frac{1}{4}, \frac{1}{2}]$ $\quad [\frac{1}{2}, \frac{3}{4}]$ $\quad [\frac{3}{4}, 1]$

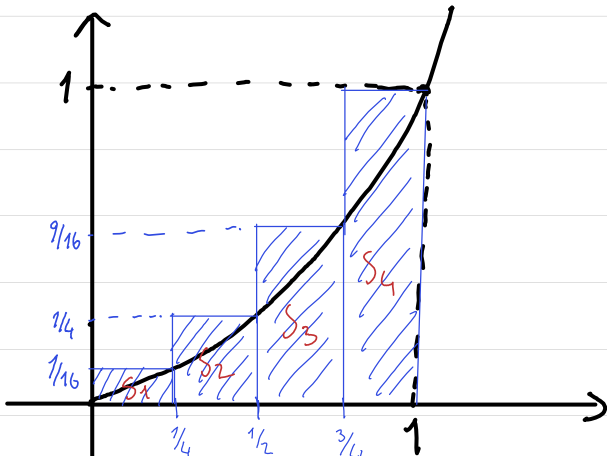
  and the rest is similar

- In midpoint case:

  we use midpoints from each part, namely,

  $[0, \frac{1}{4}] \to \frac{1}{8}$ $\quad [\frac{1}{4}, \frac{1}{2}] \to \frac{3}{8}$ $\quad [\frac{1}{2}, \frac{3}{4}] \to \frac{5}{8}$ $\quad [\frac{3}{4}, 1] \to \frac{7}{8}$

Each of three version will give the same area when we're done (we have not finished yet), so let's keep right endpoint method.



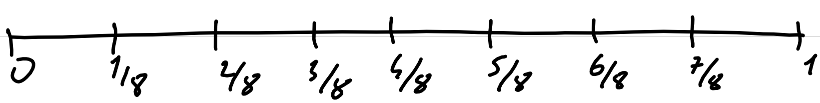
You might think "the sum of rectangles exceeds the area under the curve, so how this gives the answer?"

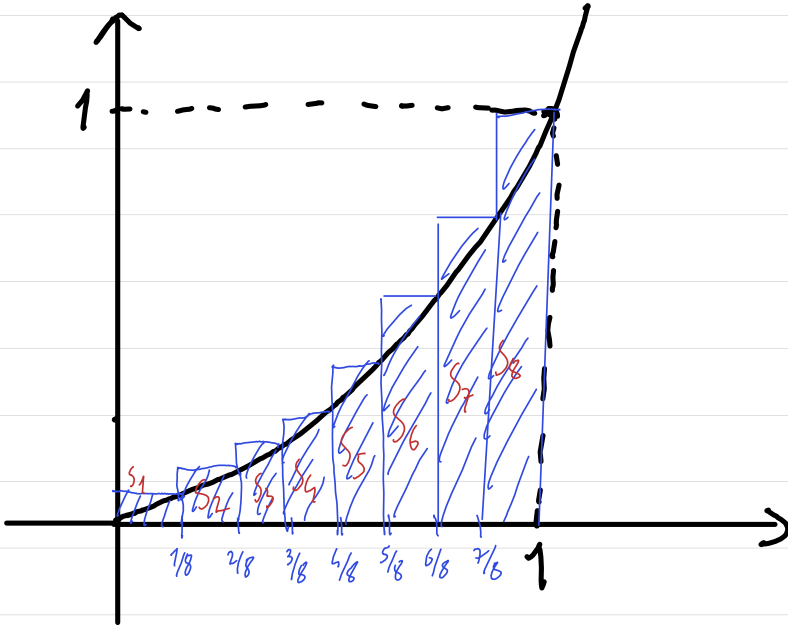
And yes, the area of these rectangles is

Let's call it

$$R_4 = \overbrace{\frac{1}{4}\cdot\frac{1}{6}}^{S_1} + \overbrace{\frac{1}{4}\cdot\frac{1}{4}}^{S_2} + \overbrace{\frac{1}{4}\cdot\frac{9}{16}}^{S_3} + \overbrace{\frac{1}{4}\cdot 1}^{S_4} = \frac{15}{32} = 0.46875$$

and this area is bigger than S. Let's see what happens if we divide the domain $[0,1]$ into <u>smaller</u> pieces like





Now we get

$$R_8 = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8 = 0.3984375.$$

This is smaller than previous case, but still bigger area than original S. Therefore, it's natural to think that if we increase the number of parts in the domain, namely, we divide $[0,1]$ into n equal pieces for bigger n, then we get an area which is closer to the actual S.

If we divide $[0,1]$ into n pieces, we get n rectangles and the sum of their areas is

$$R_n = \frac{1}{n} F\left(\frac{1}{n}\right) + \frac{1}{n} F\left(\frac{2}{n}\right) + \frac{1}{n} F\left(\frac{3}{n}\right) + \cdots$$

$$\cdots + \frac{1}{n} F\left(\frac{n-1}{n}\right) + \frac{1}{n} F\left(\frac{n}{n}\right)$$

Recall, $F(x) = x^2$, so

$$R_n = \frac{1}{n}\left(\frac{1}{n^2}\right) + \frac{1}{n}\left(\frac{4}{n^2}\right) + \frac{1}{n}\left(\frac{9}{n^2}\right) + \cdots + \frac{1}{n}\left(\frac{(n-1)^2}{n^2}\right) + \frac{1}{n}\left(\frac{n^2}{n^2}\right)$$

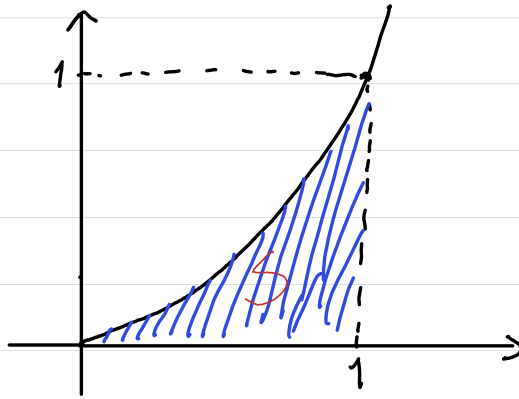
$$\boxed{R_n = \frac{1}{n^3}\left(1 + 4 + 9 + \cdots + (n-1)^2 + n^2\right)}$$

Now, recall the idea that if n is bigger, then R_n gets closer to the actual area S. Thus, we basically take the limit when $n \to \infty$ to achieve S. In other words

$$\lim_{n\to\infty} R_n = S.$$

We will learn later but, $\lim_{n\to\infty} R_n = \frac{1}{3}$.

So the area under the curve $y=x^2$ on $[0,1]$ is $\frac{1}{3}$, namely, $S=\frac{1}{3}$.



If we use left endpoints (the sum will be called L_n in this case) or midpoints (the sum will be called M_n in this case), we achieve the same result.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n = \frac{1}{3} = S.$$

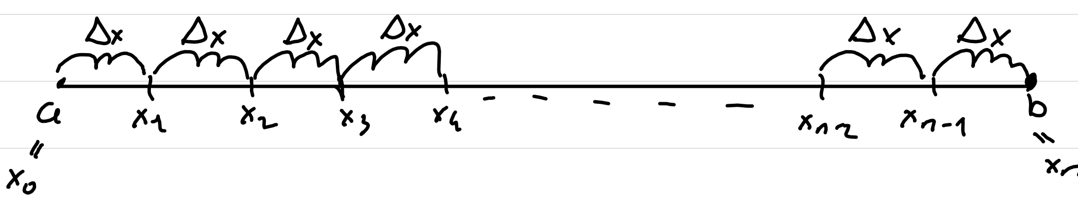
Remarks We generally call R_n as <u>upper sum</u> and L_n as <u>lower sum</u>. We'll use both more in upcoming lectures.

Let's obtain the case for arbitrary curve $y = f(x)$ on $[a, b]$.

- Divide $[a,b]$ into n pieces. Each piece has length $\Delta x = \dfrac{b-a}{n}$

this is new notation and read as "delta x"

So we have point



- Write the upper sum (i.e. use right endpoints) as follows

$$\boxed{R_n = \Delta x\big(f(x_1)\big) + \Delta x\big(f(x_2)\big) + \cdots + \Delta x\big(f(x_n)\big)}$$

- If the area under the curve $y = f(x)$ on $[a, b]$ is A, then it means

$$\boxed{A = \lim_{n \to \infty} R_n.}$$

We'll see many examples and applications of this idea in upcoming lectures.

Definite Integrals

Recall that the area A of the region that lies under the graph of continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]$$

where for $[a,b]$ being domain of f,

$\Delta x = \frac{b-a}{n}$, we have smaller intervals

$$[a=x_0, x_1], [x_1, x_2], \ldots, [x_{n-2}, x_{n-1}], [x_{n-1}, x_n = b].$$

So $x_1, x_2, \ldots, x_n$ are right endpoints for each interval. Note that we are taking

$$f(x_i)\Delta x$$

as the area of each smaller rectangle.

And don't forget, we can pick left endpoints

$x_0, x_1, \ldots, x_{n-1}$,

or midpoints

$$\frac{x_1 - x_0}{2}, \frac{x_2 - x_1}{2}, \ldots, \frac{x_n - x_{n-1}}{2}.$$

To work with this method easily, first, we introduce a notation called "sigma notation".

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

In other words, we write the sum of many numbers using a single notation with an index "i".

Final remark before giving the definition of integral:

After we divide the domain $[a,b]$ into n equal pieces and obtain the points

we're not obliged to pick only right/left/midpoints. We can arbitrary "sample" points x_i^* in each interval $[x_{i-1}, x_i]$. Thus right/left/midpoints are just special cases of sample points.

$x_i^* = x_i$ means right endpoints

$x_i^* = x_{i-1}$ means left endpoints

$x_i^* = \dfrac{x_i - x_{i-1}}{2}$ means midpoints.

If you want to play with these constructions more, look

www.desmos.com/calculator/xmn4gmutpr

You can change the function, domain, the number of intervals, sample point method, etc. and see what happens.

The whole method is called "Riemann Sum" named after mathematician Bernhard Riemann.

You can also read Riemann sum wikipedia page.

Here is the main definition:

Let f be a function on $[a,b]$. We divide it into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, \ldots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these intervals. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. In this case, we say that

f is integrable on $[a,b]$.

The two quick and important results are:

① if f is continuous on $[a,b]$, or has only finitely many jump discontinuity on $[a,b]$, then f is integrable on $[a,b]$.

② if f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$, $x_i^r = a + i\cdot\Delta x$.

In other words, ① says all essential functions are integrable, and ② says that it is enough to compute it via right endpoint method.

Later, we'll see an easier way to find $\int_a^b f(x)\,dx$.

Examples Express the limit as a definite integral on the given interval.

ⓐ $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{e^{x_i}}{1+x_i}\,\Delta x$, $[0,1]$

Considering the definition $\sum_{i=1}^{n} f(x_i)\Delta x$, we must have

$$f(x_i) = \frac{e^{x_i}}{1+x_i}.$$

So the function we have is $f(x) = \frac{e^x}{1+x}$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1+x_i} \Delta x = \int_0^1 \frac{e^x}{1+x}\,dx$$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4(x_i^*)]\Delta x$, $[2,7]$

considering the definition $\sum_{i=1}^{n} f(x_i^*)\Delta x$, we must have

$$f(x_i^*) = 5(x_i^*)^3 - 4(x_i^*),$$ so

$$\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4(x_i^*)]\Delta x = \int_2^7 (5x^3 - 4x)\,dx$$

Exercise (c) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i\sqrt{1+x_i^3}\right)\Delta x$, $[2,5]$

Without any new tool, we should compute the limits

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

to find the integrals. During upcoming lectures, we'll refer the following rules involving sums:

- $1+2+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+3^3+\cdots+n^3 = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\underbrace{c+c+\cdots+c}_{n \text{ times}} = \sum_{i=1}^{n} c = nc$
- $ca_1+ca_2+\cdots+ca_n = c(a_1+\cdots+a_n)$ namely

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

- $\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

<u>Examples</u> First, we evaluate the Riemann sum for $f(x) = x^2 + x$ using right endpoints, $a=0$, $b=3$, $n=6$. Then we also find $\int_0^3 (x^2+x)\,dx$.

$\Delta x = \frac{3-0}{6} = \frac{1}{2}$, so the divided intervals are

$[0, 0.5]$ $[0.5, 1]$ $[1, 1.5]$ $[1.5, 2]$ $[2, 2.5]$ $[2.5, 3]$

so $x_1 = 0.5$ $x_2 = 1$ $x_3 = 1.5$ $x_4 = 2$ $x_5 = 2.5$ $x_6 = 3$

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(0.5)\tfrac{1}{2} + f(1)\tfrac{1}{2} + f(1.5)\tfrac{1}{2} + f(2)\tfrac{1}{2} + f(2.5)\tfrac{1}{2} + f(3)\tfrac{1}{2}$$

$$= \frac{1}{2}(0.75 + 2 + 3.75 + 6 + 8.75 + 12)$$

$$= \frac{1}{2}(33.25) = 16.675.$$

we'll apply the definition and rules.

Now $\int_0^3 (x^2+x)dx$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{3}{n}$$

$$x_i = 0 + i\Delta x = \frac{3i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + \frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{27i^2}{n^2} + \sum_{i=1}^{n} \frac{9i}{n^2}$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{9}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty} \frac{54n^3 + \cdots}{6n^3} + \frac{9n^2+9n}{2n^2}$$

$$= \frac{54}{6} + \frac{9}{2} = 9 + \frac{9}{2} = \frac{27}{2} = 13.5$$

$$\boxed{\int_0^3 (x^2+x)dx = 13.5}$$

Properties of Definite Integrals

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b c\,dx = c(b-a)$ where c is any constant
- $\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant.
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
- If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$
- If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
- If $m \leq f(x) \leq M$ on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Examples Recall $\int_0^1 x^2 dx = \frac{1}{3}$.

Find $\int_0^1 (4+3x^2)dx$

$$\int_0^1 (4+3x^2)dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

$$= 4(1-0) + 3\left(\frac{1}{3}\right)$$

$$= 4+1 = 5$$

Examples Suppose $\int_0^{10} f(x)dx = 10$, $\int_0^8 f(x)dx = 6$.

Find $\int_8^{10} f(x)dx$.

$$\int_8^{10} f(x)dx = \int_8^0 f(x)dx + \int_0^{10} f(x)dx$$

$$= -\int_0^8 f(x)dx + \int_0^{10} f(x)dx = -6+10 = 4$$

Example: Use the properties of integrals to verify

$$2 \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le 2\sqrt{2}.$$

On $[-1,1]$, it is easy to see that

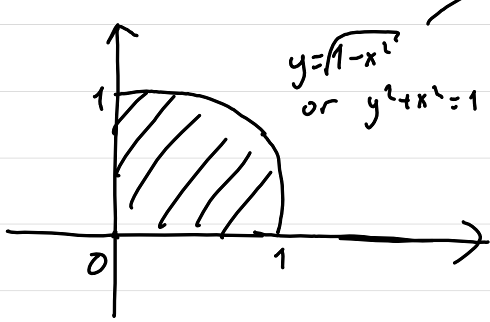
$$\underset{\text{min}}{1 = \sqrt{1+0^2}} \le \sqrt{1+x^2} \le \underset{\text{max}}{\sqrt{1+1^2} = \sqrt{2}}$$

Since $1 \le \sqrt{1+x^2} \le \sqrt{2}$ on $[-1,1]$, we get

$$\underbrace{1\,(1-(-1))}_{=\,2} \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le \underbrace{\sqrt{2}\,(1-(-1))}_{=\,2\sqrt{2}}$$

Example: Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$ by interpreting

in terms of area:

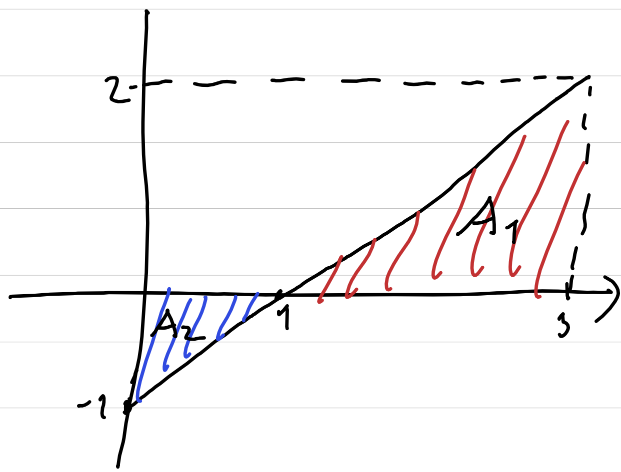


This is a quarter circle with radius 1.

The area is

$$\frac{\pi(1)^2}{4} = \frac{\pi}{4}$$

Example: Evaluate $\int_0^3 (x-1)\,dx$ by interpreting it in terms of an area.

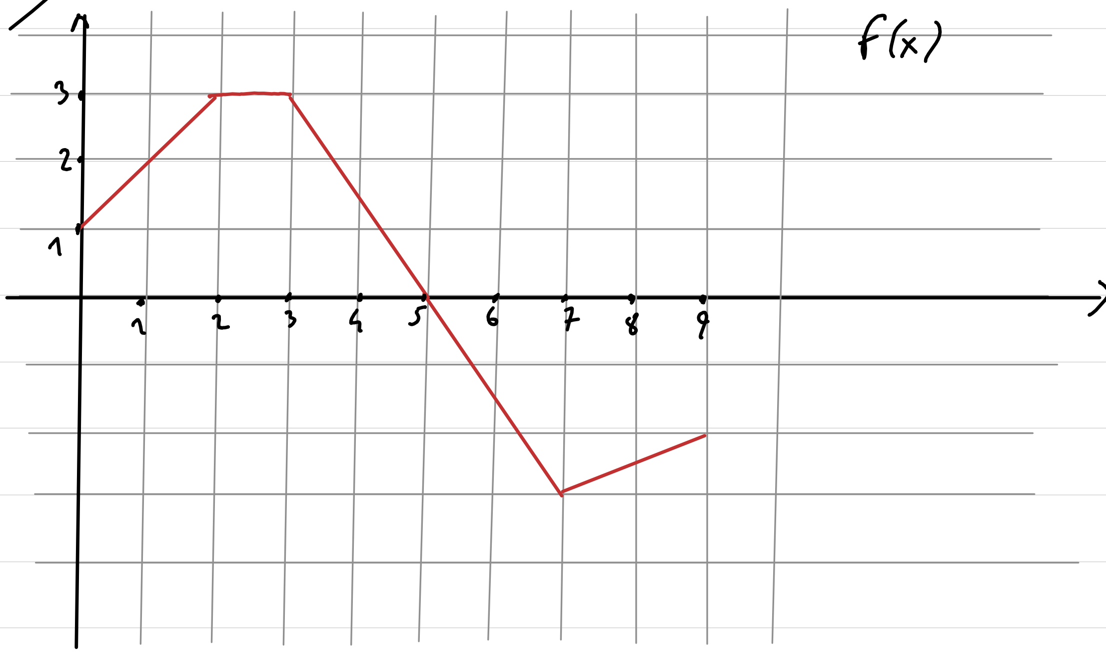


Since $y=x-1$ is a line, we have just rectangles A_1 and A_2. Then the integral is the difference of these areas

$$\int_0^3 (x-1)\,dx = A_1 - A_2 = \frac{1}{2}(2\cdot 2) - \frac{1}{2}(1\cdot 1) = 2 - 0.5 = 1.5.$$

Remark: A definite integral can be interpreted as a <u>net area</u>.

Example



! Each square has 1 unit area.

- $\int_0^2 f(x)\,dx = 4$
- $\int_5^7 f(x)\,dx = -3$
- $\int_0^5 f(x)\,dx = 10$
- $\int_0^8 f(x)\,dx = 10 - 8 = 2$

We use the squares and triangles to find the integrals (net areas).

Fundamental Theorem of Calculus (FTC)

Suppose f is continuous on $[a,b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We can write 1 as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

and 2 as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Taken together, the two parts of the FTC say that differentiation and integration are inverse processes.

We'll omit their proofs, but intuitively, you can consider

$$\frac{g(x+h)-g(x)}{h} \approx f(x) \quad \text{for 1. part}$$

and use 1. part to prove 2. part since g is already an antiderivative of f, so we have $F = g + C$ for an arbitrary constant C.

We focus more on examples.

Examples Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\, dt$.

Since $f(t) = \sqrt{1+t^2}$ is continuous, by FTC1, we get $g'(x) = \sqrt{1+x^2}$

Example Find $\frac{d}{dx}\left(\int_1^{x^4} \sec t \, dt \right)$.

Here, we should be careful about x^4. We can apply Chain Rule together with FTC1.

Let $x^4 = u$.

Then $\frac{d}{dx}\left(\int_1^{x^4} \sec t\, dt\right)$

$$= \frac{d}{dx}\left(\int_1^{u} \sec t\, dt\right)$$

$$= \frac{d}{du}\left(\int_1^{u} \sec t\, dt\right)\frac{du}{dx} \quad \text{by chain rule}$$

$$= \sec u \cdot \frac{du}{dx} \quad \text{by FTC1}$$

$$= (\sec x^4)\cdot 4x^3$$

Example: Evaluate the integral $\int_1^3 e^x\,dx$.

Since $F(x) = e^x$ is antiderivative of e^x, using FTC2 we get

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e.$$

Remark: We generally use the notation

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Examples

- $$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = 1/3$$

  $\frac{x^3}{3}$ is antiderivative of x^2

- $$\int_3^6 \frac{dx}{x} = \ln|x|\Big|_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2.$$

  $\ln|x|$ is antiderivative of $\frac{1}{x}$

- $$\int_0^{\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

  $\sin x$ is antiderivative of $\cos x$.

Remark You may think, for example,

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big|_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

but this is wrong since $\frac{1}{x^2}$ is not continuous on $[-1,3]$.

Example: Find the derivative of the function

$$R(y) = \int_y^2 t^3 \sin t \, dt.$$

We can use the properties of integral and write

$$R(y) = -\int_2^y t^3 \sin t \, dt.$$

So by FTC 1,

$$R'(y) = -(y^3 \sin y)$$

Example: $\int_0^1 (x^e + e^x)\,dx = \frac{x^{e+1}}{e+1} + e^x \Big|_0^1 = \left(\frac{1}{e+1} + e\right) - (0+1)$

$$= \frac{1}{e+1} + e - 1$$

$\frac{x^{e+1}}{e+1} + e^x$ is antiderivative of $x^e + e^x$

Example: Find the derivative of

$$y = \int_{\sin x}^1 \sqrt{1+t^2} \, dt.$$

First, we have

$$y = -\int_1^{\sin x} \sqrt{1+t^2}\, dt.$$

Use $u = \sin x$

$$y = -\int_1^{u} \sqrt{1+t^2}\, dt$$

By chain rule

$$\frac{dy}{dx} = -\frac{d}{du}\left(\int_1^{u} \sqrt{1+t^2}\, dt\right)\frac{du}{dx}$$

By FTC 1

$$\frac{d}{dx} y = -\sqrt{1+u^2}\,\frac{du}{dx}$$

$$= -\sqrt{1+\sin^2 x}\cdot \cos x$$

Indefinite Integral

This is not a new topic, actually. In definite integral, by FTC, we obtained

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is the antiderivative of f, namely $F' = f$. We'll omit boundaries on the integral and consider

$$\int f(x)dx.$$

This is called indefinite integral. Instead of saying

"F is antiderivative of f",

we'll say

"The indefinite integral of f is F"

and write

$$\int f(x)dx = F(x)$$

Examples

- $\int x^2 dx = \frac{x^3}{3} + C$

- $\int e^x dx = e^x + C$

- $\int \frac{1}{x} dx = \ln|x| + C$

To solve $\int f(x)dx$, we just find the antiderivative of f. DON'T FORGET to put the constant C.
We'll mostly use the following integrals and rules in practice:

1. $\int c\, f(x)\, dx = c \int f(x)\, dx$

2. $\int f(x) \pm g(x)\, dx = \int f(x) dx \pm \int g(x) dx$

3. $\int k\, dx = kx + C$ (Note that $f(x)=k$ constant function here)

4. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq 1)$

5. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

6. $\displaystyle\int e^x\,dx = e^x + C$

7. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} + C$

8. $\displaystyle\int \sin x\,dx = -\cos x + C$

9. $\displaystyle\int \cos x\,dx = \sin x + C$

10. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

11. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

Remark: Rule 4 can be also used for root functions. For example

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{3/2} + C$$

Examples

- $\displaystyle\int \frac{1+\sqrt{x}+x}{x}\,dx$

$$= \int \frac{1}{x} + \frac{1}{\sqrt{x}} + 1 \, dx$$

$$= \int \frac{1}{x}\,dx + \int x^{-1/2}\,dx + \int 1\,dx$$

$$= \ln|x| + \frac{x^{1/2}}{1/2} + x + C$$

- $\displaystyle\int 2^t(1+5^t)\,dt$

$$= \int (2^t + 10^t)\,dt$$

$$= \int 2^t\,dt + \int 10^t\,dt = \frac{2^t}{\ln 2} + \frac{10^t}{\ln 10} + C$$

- $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}\,dx = \int 2\cos x\,dx$

$$= 2\sin x + C$$

- $\displaystyle\int_0^{\pi} (5e^x + 3\sin x)\,dx = 5e^x - 3\cos x \Big|_0^{\pi}$

$$= (5e^{\pi} - 3\cos\pi) - (5e^0 - 3\cos 0)$$
$$= 5e^{\pi} + 3 - 5 + 3 = 5e^{\pi} + 1$$

We used FTC, $5e^x - 3\cos x$ is antiderivative of $5e^x + 3\sin x$. We omit $+C$ factor since it will be cancelled in computation

The Substitution Rule

With this rule, we aim to solve more complex integrals. For example, since we don't know directly antiderivative of $2x\sqrt{1+x^2}$, we don't have enough tool to solve

$$\int 2x\sqrt{1+x^2}\,dx. \qquad \emptyset$$

The rule will simplify the expression.

First, we substitute $1+x^2 = u$, namely, instead of $1+x^2$, we write u. But if we take the derivative of both sides, we get

$$2x = \frac{du}{dx} \quad \text{i.e.} \quad 2x\,dx = du.$$

So we can write $\emptyset$ as

$$\int \sqrt{u}\,du$$

But it's very easy to solve:

$$\int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C = \frac{(1+x^2)^{3/2}}{3/2} + C.$$

Therefore, the substitution makes computation easier. When the expression is complex, we can change some expression "$f(x)$" with "u" and achieve more doable integrals.

In general case, if we have integrals like

$$\int f(g(x)) \cdot g'(x)\, dx$$

we can use this method, namely,

we substitute $g(x) = u$,

then $g'(x) = \frac{du}{dx}$ i.e. $g'(x)dx = du$, so

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

Examples $\int x^2 e^{x^3} dx$

substitute $x^3 = u$, then $3x^2 = \frac{du}{dx}$

so $x^2 dx = \frac{du}{3}$.

$$\int x^2 e^{x^3}dx = \int \frac{e^u}{3}\,du = \frac{e^u}{3} + C = \frac{e^{x^3}}{3} + C$$

- $\int \frac{dx}{5-3x}$

substitute $5-3x=u$, then $-3=\frac{du}{dx}$, so $dx=\frac{du}{-3}$

$$\int \frac{dx}{5-3x} = \int \frac{1}{u}\frac{du}{-3} = \int \frac{du}{-3u} = -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + C$$

$$= -\frac{1}{3}\ln|5-3x| + C$$

- $\int \cos^3\theta \sin\theta \, d\theta$

substitute $\cos\theta = u$, then $-\sin\theta = \frac{du}{d\theta}$, so $\sin\theta\, d\theta = -du$

$$\int -u^3 du = \frac{-u^4}{4} + C = -\frac{\cos^4\theta}{4} + C$$

• $\int \frac{\sec^2 x}{\tan^2 x}\, dx$

substitue $\tan x = u$, then $(\sec^2 x)dx = du$

$$\int \frac{du}{u^2} = \int u^{-2} du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

$$= -\frac{1}{\tan x} + C$$

$$= -\cot x + C$$

We can use the substitution rule also for definite integrals, but in this case, we should also change endpoints according to the substitution.

Examples $\int_0^4 \sqrt{2x+1}\, dx$,

Substitute $u = 2x+1$ ⊛, then $\frac{du}{dx} = 2$, so $dx = \frac{du}{2}$.

Also if $x=0$, by ⊛, $u = 2\cdot 0 + 1 = 1$.
if $x=4$, by ⊛, $u = 2\cdot 4 + 1 = 9$.

So we have

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \frac{\sqrt{u}}{2}\,du = \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_1^9$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \quad u^{3/2}\Big|_1^9$$

$$= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{1}{3}\left(27 - 1\right) = \frac{26}{3}$$

Examples $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

substitute $\ln x = u$, then $\dfrac{1}{x} = \dfrac{du}{dx}$

so $\dfrac{dx}{x} = du$.

Also if $x=1$, then $u = \ln 1 = 0$

if $x = e$, then $u = \ln e = 1$.

So $\displaystyle\int_1^e \frac{\ln x}{x}dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

Remark: Some quick tips

Suppose f is continuous on $[-a, a]$, then

- if f is even (i.e. $f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

- if f is odd (i.e. $f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)dx = 0.$$

You can observe these from the graphs of an even and an odd functions, and that fact that the definite integral is the net area under the curve.

Examples

- $\displaystyle\int_{-5}^{5} x^3 dx = 0$ since x^3 is an odd function

- $\displaystyle\int_{-\pi}^{\pi} \cos x\, dx = 2\int_0^{\pi} \cos x\, dx$ since $\cos x$ is an even function.

Let's do one more example about definite integrals and substitution rule.

Example $\displaystyle\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\, dt$

Substitute $\cos t = u$, then $-\sin t = \frac{du}{dt}$,

so $\sin t\, dt = -du$.

Also, if $t = 0$, then $u = 1$.

if $t = \pi/6$, then $u = \frac{\sqrt{3}}{2}$

So $\displaystyle\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt = \int_1^{\sqrt{3}/2} \frac{-du}{u^2} = u^{-1}\Big|_1^{\sqrt{3}/2}$

$$= \frac{1}{\frac{\sqrt{3}}{2}} - \frac{1}{1}$$

$$= \frac{2}{\sqrt{3}} - 1$$

Exercise Evaluate the following integral.

- $\displaystyle\int e^x \sqrt{1+e^x}\,dx$

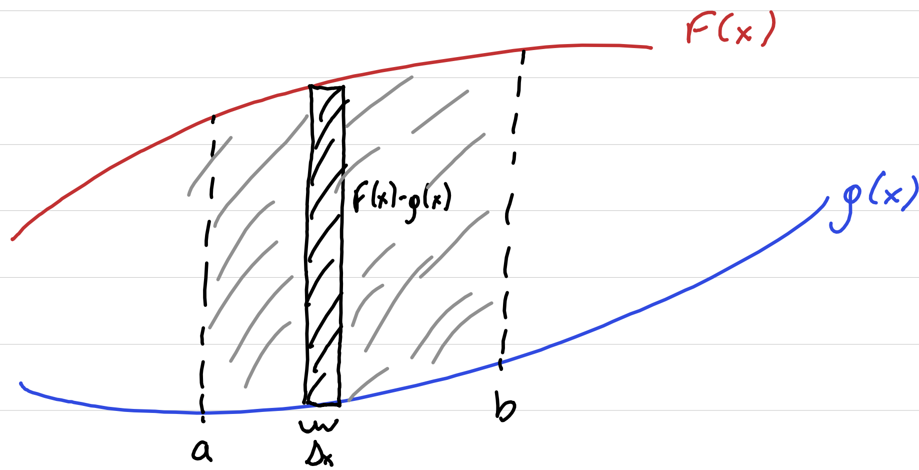
- $\displaystyle\int \frac{x}{x^2+4}\,dx$

- $\displaystyle\int_0^2 (x-1)\,e^{(x-1)^2}\,dx$

- $\displaystyle\int_0^{\pi/2} \cos x \cdot \sin(\sin x)\,dx$

Areas Between Curves

Suppose we have two function $f(x)$ and $g(x)$ and we want to find the area between their graphs



Like we did before, we divide the gray region into small rectangles, But in this case, the height is given by $f(x)-g(x)$, Thus, when we use the limit of sums of areas of these rectangles, we achieve $\int_a^b f(x)-g(x)\,dx$.

And this is the right intuition, i.e. we define the area between two curves like this:

Definition. The area A of the region bounded by the curves $y=f(x)$ and $y=g(x)$ from $x=a$ to $x=b$, where f, g are continuous and $f(x) \geqslant g(x)$ on $[a,b]$ is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Example. The area of the region bounded above by $y=e^x$, bounded below by $y=x$, from $x=0$ to $x=1$ is

$$\int_0^1 (e^x - x)\, dx = e^x - \frac{x^2}{2} \Big|_0^1$$

$$= \left(e - \frac{1}{2}\right) - (1-0) = e - \frac{3}{2}$$

Remark: In some cases, we'll ask only

"Find the area between $y=f(x)$ and $y=g(x)$".

Namely, we will not give endpoints. In this case we mean that there are some intersection points between two curves and we use these as endpoints.

Example: Find the area of the region enclosed by x^2 and $2x-x^2$.

Solution: First solve when

$$x^2 = 2x - x^2$$

It means

$$0 = 2x - 2x^2 = 2x(1-x),$$

so $x=0$ or $x=1$.

So we mean the area from $x=0$ to $x=1$.

NOTE that we also be careful about the order of the function. Since $2x-x^2$ is above x^2 on $[0,1]$, we have

$$\boxed{\int_0^1 [(2x-x^2)-x^2]\,dx}$$

!!! NOT $\int_0^1 [x^2-(2x-x^2)]\,dx$.

So the area enclosed by $2x-x^2$ and x^2 is

$$\int_0^1 [2x-x^2-x^2]\,dx$$

$$= \int_0^1 2x-2x^2\,dx = x^2-\frac{2}{3}x^3\Big|_0^1$$

$$= \left(1-\frac{2}{3}\right)-(0-0)$$

$$= \frac{1}{3}$$

Examples Find the area enclosed by $12-x^2$ and x.

If $x = 12 - x^2$, then
$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4, 3.$$

Also, on $[-4, 3]$, we have $12 - x^2 \geq x$.

So the area is $A = \int_{-4}^{3} \left[(12 - x^2) - x\right] dx$

$$= 12x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-4}^{3}$$

$$= \left(36 - 9 - \frac{9}{2}\right) - \left(-48 + \frac{64}{3} - 8\right)$$

$$= \frac{343}{6}$$