Math 301 Sample Midterm 1

Fall 2026

Oct 2026

Last name: ______________________________ First name: ______________________________

Instructions

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy immediately.
- The actual exam will have **5 questions** (100 points, 50 minutes) with the same structure as Questions 1 to 5 of this sample: True/False, a system with parameters, one application problem, a linear transformation, and matrix algebra.
- Question 3 contains **four** application problems. Exactly **one problem type** of one of these four types will appear on the exam. Prepare for all four.
- Do not use books, calculators, computers, tablets, or phones.
- You may use a letter-size, one-sided cheat sheet but no other notes.
- Write legibly and cross out any work that you do not wish to have scored.
- Show all of your work unless stated otherwise. Unsupported answers may not earn credit.

1. (20 points) Read the sentences below carefully. Examine the propositions below and determine whether they are true or false. You do not need to provide an explanation, just mark your choice. (Each is 4 points)

True — False A homogeneous system $A\mathbf{x} = \mathbf{0}$ with more unknowns than equations always has a nontrivial solution.

True — False If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^3$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

True — False Let M be the transition matrix of a migration model and $\mathbf{x}_0$ the initial population vector. Then the population vector after two years is $2M\mathbf{x}_0$.

True — False Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, that is, $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A.

True — False If A is an invertible $n \times n$ matrix and $AB = AC$, then $B = C$.

2. Consider the system of equations

$$\begin{aligned} x - y + 2z &= 2 \\ 2x - y + 5z &= 5 \\ x + y + (k+1)z &= l \end{aligned}$$

(a) (3 points) Write the augmented matrix of the system.

(b) (5 points) Reduce the augmented matrix to **row-echelon form**.

(c) (3 points) Find all values of k and l (if any) where the system has no solutions.

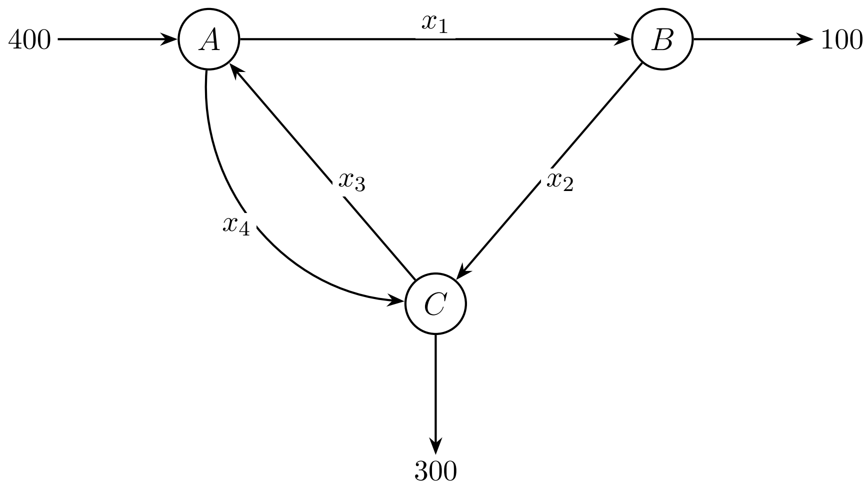
(d) (3 points) Find all values of k and l (if any) where the system has a unique solution.

(e) (3 points) Find all values of k and l (if any) where the system has an infinite number of solutions.

(f) (3 points) For $k = 3$ and $l = 4$, write the general solution in **parametric vector form**.

3. **Applications.** Below are four application problems, one of each type covered in the course so far. On the actual exam, **exactly one** problem of one of these four types will appear, worth 20 points.

   (a) (20 points) **(Network flow)** The figure shows the traffic flow (in vehicles per hour) on a network of one-way streets. The variables $x_1, \ldots, x_4$ denote the unknown flows on the inner streets. At each intersection the total flow in equals the total flow out.



      i. (6 points) Write the linear system that describes the flows at the three intersections A, B, C.

      ii. (8 points) Find the general solution of the system.

      iii. (6 points) Suppose the street from C to A is closed, so $x_3 = 0$. If every flow must be nonnegative, find the largest possible value of x_4.

(b) (20 points) **(Chemical equation)** Iron is produced from iron ore according to the unbalanced equation

$$\mathrm{Fe_2O_3} + \mathrm{CO} \longrightarrow \mathrm{Fe} + \mathrm{CO_2}.$$

i. (6 points) Let x_1, x_2, x_3, x_4 be the numbers of molecules of Fe_2O_3, CO, Fe, CO_2. Using vectors in $\mathbb{R}^3$ whose entries count the atoms of Fe, O, C, write a vector equation that expresses that the equation is balanced, and the corresponding homogeneous linear system.

ii. (10 points) Row reduce the coefficient matrix and find the general solution of the system.

iii. (4 points) Balance the chemical equation using the smallest possible positive whole numbers.

(c) (20 points) **(Exchange economy)** An economy has three sectors: Agriculture, Manufacturing, and Services. Each sector sells its total output to the three sectors according to the table below (each column shows how the output of one sector is distributed).

	Distribution of output from:		
Purchased by:	Agriculture	Manufacturing	Services
Agriculture	0.1	0.3	0.1
Manufacturing	0.1	0.2	0.5
Services	0.8	0.5	0.4

Let p_A, p_M, p_S denote the prices (values) of the total outputs of the three sectors. Equilibrium prices are prices for which each sector's income equals its expenses.

i. (6 points) Write the linear system that the equilibrium prices must satisfy.

ii. (10 points) Solve the system and give the general solution.

iii. (4 points) If the output of Services is priced at 300 (million dollars), what are the equilibrium prices of Agriculture and Manufacturing?

(d) (20 points) **(Transition matrix)** Two supermarkets, X and Y, compete for the same 5000 customers. Each month, 90% of X's customers keep shopping at X and 10% switch to Y; 30% of Y's customers switch to X and 70% stay at Y. This month X has 3000 customers and Y has 2000.

i. (5 points) Using the order (X, Y), write the transition matrix M and the vector $\mathbf{x}_0$ so that $\mathbf{x}_k = M\mathbf{x}_{k-1}$ gives the numbers after k months. Check that each column of M adds up to 1.

ii. (5 points) Find $\mathbf{x}_1$ and $\mathbf{x}_2$.

iii. (5 points) Is M invertible? Is the transformation $\mathbf{x} \mapsto M\mathbf{x}$ one-to-one? Explain.

iv. (5 points) How many customers did each supermarket have **one month ago**? (Find $\mathbf{x}_{-1}$ with $M\mathbf{x}_{-1} = \mathbf{x}_0$.)

4. (a) (6 points) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x_1, x_2) = (x_1 + x_2,\ x_1x_2)$. Is S a linear transformation? Justify your answer.

(b) (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that **first** rotates each point by $90°$ counterclockwise about the origin and **then** reflects the result through the x_1-axis. Find $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and the standard matrix A of T.

(c) (4 points) Is T one-to-one? Is T onto $\mathbb{R}^2$? Justify using A.

(d) (4 points) Describe T as a single geometric transformation of the plane.

5. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) (4 points) Compute $2A - B^T$, AB, and BA.

(b) (10 points) Find P^{-1} by row reducing $[\,P \mid I\,]$.

(c) (3 points) Use P^{-1} to solve $P\mathbf{x} = \mathbf{b}$.

(d) (3 points) Without any further computation, decide whether the columns of P are linearly independent, and whether the linear transformation $\mathbf{x} \mapsto P\mathbf{x}$ is onto $\mathbb{R}^3$. Explain.

Extra practice. The two questions below are not part of the 5-question exam structure, but the same ideas (linear independence, span, pivot counting) can appear inside the True/False question and inside Questions 2, 4 and 5.

6. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ h \end{bmatrix}.$$

(a) (8 points) For which value(s) of h is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent?

(b) (4 points) For that value of h, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(c) (4 points) For $h = 0$, do the vectors span $\mathbb{R}^3$? Explain.

(d) (4 points) For $h = -4$, is the vector $\mathbf{b} = (0, 0, 1)$ in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Explain.

7. (20 points) Fill the following table. If possible, use the quick ways to determine the condition being asked. If you do need calculations, you can use the next page for scratch work. Please provide explanation for your answers. (Each row is 5 points.)

Set of Vectors	**Determine whether the vectors are linearly independent or not.**	**Determine whether the vectors span $\mathbb{R}^3$ or not.**
$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$		
Explanation		
$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$		
Explanation		
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$		
Explanation		
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$		
Explanation		

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.