Math 301 Sample Midterm 2

Fall 2026

Nov 2026

Last name: _____ First name: _____

Instructions

- This examination consists of 15 pages, not including this cover page. Verify that your copy of this examination contains all 15 pages. If your examination is missing any pages, then obtain a new copy immediately.
- The actual exam will have **5 questions** (100 points, 50 minutes) with the same structure as Questions 1 to 5 of this sample: True/False (5 statements), determinants, the Leontief input-output model, subspaces and bases of a matrix, and signals and difference equations. Each question is worth 20 points.
- The exam covers Weeks 6 to 11: Sections 2.6, 2.7, 3.1 to 3.3, 4.1 to 4.5, 4.7 and 4.8 (the Leontief model, computer graphics, determinants, vector spaces, bases, dimension and rank, signals, difference equations).
- Questions 6 to 8 are **extra practice**. They are not part of the 5-question structure, but the same ideas can appear inside Questions 1 to 5.
- You may use a handheld calculator. The calculator of a phone is **not** allowed. Do not use books, computers, tablets, or phones.
- You may use a letter-size, two-sided cheat sheet but no other notes.
- Write legibly and cross out any work that you do not wish to have scored.
- Show all of your work unless stated otherwise. Unsupported answers may not earn credit.
- Two parts are marked **(Challenge)**. They are harder than the rest and are worth 5 points each. You may skip them and return to them if time remains.

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1. (20 points) Read the sentences below carefully. Examine the propositions below and determine whether they are true or false. You do not need to provide an explanation, just mark your choice. (Each is 4 points)

   True — False The translation $(x, y) \mapsto (x + 2,\ y - 1)$ is a linear transformation of $\mathbb{R}^2$.

   True — False If A and B are $n \times n$ matrices, then $\det(A + B) = \det A + \det B$.

   True — False If A is an invertible $n \times n$ matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

   True — False The set of all symmetric 2×2 matrices (matrices with $A^T = A$) is a subspace of $M_{2\times 2}$.

   True — False If A is a 4×6 matrix with $\operatorname{rank} A = 4$, then $\operatorname{Col} A = \mathbb{R}^4$.

2. Let

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}.$$

(a) (6 points) Compute $\det A$ by cofactor expansion. At each step choose a row or column that involves the least computation, say which one you chose, and write the sign of every position you use.

(b) (4 points) Using your answer to (a) and the properties of determinants (do not compute any new determinant from scratch), find

$$\det(A^{-1}), \qquad \det(A^T A), \qquad \text{and} \qquad \det B,$$

where B is obtained from A by the row operations $R_1 \leftarrow R_1 - 2R_3$ and then $R_2 \leftrightarrow R_4$.

(c) (5 points) Use Cramer's rule to find **only** x_3 in the system below. Do not solve for x_1 or x_2.

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_2 + x_3 &= 3 \\ x_1 + 3x_3 &= -2 \end{aligned}$$

(d) (5 points) **(Challenge)** Let A be a 3×3 matrix with $A^T = -A$. Show that $\det A = 0$.

3. **(Leontief input-output model.)** An economy has two sectors, Agriculture and Services. For each unit of output, Agriculture uses .3 unit of agricultural products and .2 unit of services. For each unit of output, Services uses .1 unit of agricultural products and .4 unit of services. (All units are in millions of dollars.)

   (a) (4 points) Using the order (Agriculture, Services), write the consumption matrix C. Check that each column sum of C is less than 1, and say what Theorem 11 then guarantees.

   (b) (4 points) If Agriculture produces 100 units and Services produces 50 units, what intermediate demand is created?

   (c) (6 points) Write the matrix $I - C$ and compute $(I - C)^{-1}$.

   (d) (6 points) Find the production level that satisfies a final demand of 20 units of agricultural products and 40 units of services.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 7 \\ 1 & -1 & 3 & -4 \end{bmatrix}.$$

(a) (5 points) Row reduce A to **reduced** echelon form. Write each row operation you use.

(b) (4 points) Give a basis for $\operatorname{Col} A$. Explain why the vectors you chose form a basis.

(c) (4 points) Give a basis for $\operatorname{Nul} A$.

(d) (3 points) State $\operatorname{rank} A$, $\operatorname{nullity} A$ and $\dim \operatorname{Row} A$. Is $\operatorname{Col} A = \mathbb{R}^3$? Explain.

(e) (4 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be the basis for $\operatorname{Col} A$ from (b), in that order. One of the vectors $\mathbf{w} = (3, 8, 1)$ and $\mathbf{y} = (3, 8, -1)$ is in $\operatorname{Col} A$ and the other is not. Decide which is which, and find the coordinate vector of the one that is in $\operatorname{Col} A$ relative to $\mathcal{B}$.

5. Let $\mathbb{S}$ be the space of signals. Parts (b) to (d) concern the difference equation

$$y_{k+2} - 6y_{k+1} + 8y_k = 0 \qquad \text{for all } k. \tag{$*$}$$

(a) (3 points) Let $T : \mathbb{S} \to \mathbb{S}$ be the linear time invariant transformation $T(\{x_k\}) = \{x_k + 3x_{k-1}\}$, and let δ be the delta signal ($d_0 = 1$, $d_k = 0$ for $k \neq 0$). Compute the entries of $T(\delta)$ at $k = -1, 0, 1, 2$, and find a nonzero signal in the kernel of T.

(b) (7 points) Find two solutions of $(*)$ of the form $\{r^k\}$. Explain why they form a basis for the solution space of $(*)$, and write the general solution.

(c) (5 points) Find the solution of $(*)$ with $y_0 = 3$ and $y_1 = 10$.

(d) (5 points) **(Challenge)** Now consider the **nonhomogeneous** equation

$$y_{k+2} - 6y_{k+1} + 8y_k = 9 \qquad \text{for all } k.$$

Find its general solution, and then the solution with $y_0 = 5$, $y_1 = 9$.

Extra practice. Questions 6 to 8 are not part of the 5-question exam structure, but the same ideas (subspace tests, the Rank Theorem, LTI transformations, the Casorati test, parameters in determinants, the adjoint, areas and volumes, homogeneous coordinates, data matrices, coordinates, dimension of a span, first-order systems) can appear inside the True/False question and inside Questions 2 to 5.

6. **Short questions on subspaces, rank and signals.**

(a) (8 points) Decide whether each set is a subspace of the given vector space. If it is, justify it (check the three properties, or write the set as a span or as a null space). If it is not, give a specific counterexample.

i. $K = \{(x, y) : xy = 0\}$ in $\mathbb{R}^2$.

ii. $W = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \text{ real} \right\}$ in $M_{2\times 2}$.

iii. $L = \{(x, y, z) : x = y + 1\}$ in $\mathbb{R}^3$.

iv. $H = \{(a + b,\ a - b,\ 2b) : a, b \text{ real}\}$ in $\mathbb{R}^3$. If H is a subspace, also give a basis and $\dim H$.

(b) (6 points) Answer with a short explanation.

i. A is a 4×7 matrix with $\operatorname{rank} A = 4$. Find $\operatorname{nullity} A$. Is $\operatorname{Col} A = \mathbb{R}^4$?

ii. B is a 6×4 matrix with $\operatorname{nullity} B = 0$. Find $\operatorname{rank} B$ and $\dim \operatorname{Row} B$. Is $\operatorname{Col} B = \mathbb{R}^6$?

iii. Can a 3×5 matrix have nullity 1?

(c) (6 points) For each transformation $T : \mathbb{S} \to \mathbb{S}$ decide whether it is linear time invariant, linear but not time invariant, or not linear. Justify briefly.

i. $T(\{x_k\}) = \{x_{k+1} + x_{k-1}\}$

ii. $T(\{x_k\}) = \{k\, x_k\}$

iii. $T(\{x_k\}) = \{x_k + 1\}$

Then decide whether the signals $\{1\}$, $\{(-1)^k\}$ and $\{2^k\}$ are linearly independent, using the Casorati matrix at $k = 0$.

7. **More determinants.**

(a) (5 points) For which values of h is the matrix $\begin{bmatrix} 1 & 1 & h \\ 1 & h & 1 \\ h & 1 & 1 \end{bmatrix}$ invertible?

(b) (5 points) Consider the system, where s is a parameter: $s\,x_1 + 3x_2 = 1, \quad 3x_1 + s\,x_2 = 2$. Find all values of s for which the system has a unique solution, and use Cramer's rule to write x_1 and x_2 in terms of s. For the other values of s, decide (by another method) whether the system has no solution or infinitely many.

(c) (6 points) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

i. Compute all nine cofactors of A and write $\operatorname{adj} A$.

ii. Compute $A\,(\operatorname{adj} A)$ and use it to find $\det A$. Then write A^{-1}.

iii. Using cofactors only, find the $(2,3)$-entry of A^{-1} and compare with (ii).

(d) (4 points) Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,2)$, $(0,3,1)$ and $(2,1,0)$. Then let $T(\mathbf{x}) = B\mathbf{x}$ with $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$: find the volume of the image of this parallelepiped under T, and the volume of the image of the unit ball (volume $\frac{4}{3}\pi$) under T.

(e) (4 points) Let S be the parallelogram with vertices $(1,1)$, $(3,2)$, $(2,4)$ and $(4,5)$, and let $T(\mathbf{x}) = P\mathbf{x}$ with $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the area of S and the area of $T(S)$.

8. **Coordinates, dimension, and more applications.**

(a) (5 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ with $\mathbf{b}_1 = (1, 1, 0)$, $\mathbf{b}_2 = (0, 1, 1)$, $\mathbf{b}_3 = (1, 0, 2)$.

i. Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

ii. Find $[\,\mathbf{x}\,]_{\mathcal{B}}$ for $\mathbf{x} = (4, 5, 7)$.

iii. Find $\mathbf{y}$ if $[\,\mathbf{y}\,]_{\mathcal{B}} = (1, -1, 2)$.

(b) (5 points) Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ with $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 4, 2)$, $\mathbf{v}_3 = (1, 0, 3)$, $\mathbf{v}_4 = (3, 4, 5)$. Find $\dim W$ and a basis for W consisting of some of the $\mathbf{v}_i$. Write each of the other vectors as a combination of your basis vectors.

(c) (3 points) In homogeneous coordinates, the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ represents a 2D transformation. Find the image of the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 1)$ under this transformation.

(d) (2 points) Write the equation $y_{k+2} - 6y_{k+1} + 8y_k = 0$ of Question 5 as a first-order system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with $\mathbf{x}_k = (y_k,\, y_{k+1})$. Give the matrix A.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.