# M384: Logic — Question Pool for Exam 1

**How this pool is used.** The exam questions are taken from this pool. Some questions will be asked exactly as they appear here; others will appear with the nouns, the set $\Gamma$, or the sentence $\varphi$ changed. We will work through a selection in the review session. Nothing in the pool is off limits.

**Conventions.** $\Gamma$ is always a set of sentences; $\varphi, \psi$ are sentences. $\Gamma \models \varphi$ means every model of $\Gamma$ is a model of $\varphi$. $\Gamma \vdash \varphi$ means there is a proof tree in the relevant proof system with root $\varphi$ whose (non-withdrawn) leaves are in $\Gamma$ or are instances of (Axiom). For a noun $p$ we write $[\![p]\!]$ for its interpretation in a model. In $\mathcal{S}^\dagger$ the literal $\bar{p}$ is interpreted as $M \setminus [\![p]\!]$, and No $p$ are $q$ is another way of writing All $p$ are $\bar{q}$. The rule sheets for $\mathsf{A}$, $\mathsf{S}$ and $\mathsf{S}^\dagger$ are on the last pages.

**What "with a reason" means.** When you answer true/false, a reason is a model, a proof tree, or a reference to a theorem (soundness, completeness, a lemma) applied correctly. If you use soundness or completeness you must say which one you are using and why it applies.

1. Let $M = \{1,2,3,4,5,6\}$ with $[\![a]\!] = \varnothing$, $[\![b]\!] = \{1,2\}$, $[\![c]\!] = \{2,3\}$, $[\![d]\!] = \{1,2,3\}$. The language $\mathcal{A}$ over $\mathsf{P} = \{a,b,c,d\}$ has 16 sentences. List the ones that are true in this model, and explain in one sentence why every sentence whose first noun is $a$ is on your list.

2. Prove each claim directly from the definition of $\models$ (no soundness, no completeness).
   (a) All $p$ are $q$, All $q$ are $r \models$ All $p$ are $r$.
   (b) All $p$ are $q \not\models$ All $q$ are $p$.
   (c) All $p$ are $q$, All $r$ are $q \not\models$ All $p$ are $r$.
   (d) $\varnothing \models$ All $p$ are $p$.

3. Here $\Gamma = \varnothing$.
   (a) Which of the following says the same as $\varnothing \not\models \varphi$: "some model does not satisfy $\varphi$" or "every model does not satisfy $\varphi$"?
   (b) If $p = q$ show $\varnothing \vdash$ All $p$ are $q$.
   (c) If $p = q$ show $\varnothing \models$ All $p$ are $q$.
   (d) If $p \neq q$ show $\varnothing \not\models$ All $p$ are $q$.
   (e) If $p \neq q$ show $\varnothing \nvdash$ All $p$ are $q$; if you use soundness or completeness, say which.

4. Let $\Gamma = \{$All $l$ are $m$, All $q$ are $l$, All $m$ are $p$, All $n$ are $p$, All $l$ are $q\}$.
   (a) Give a proof tree showing $\Gamma \vdash$ All $q$ are $p$.
   (b) Give a proof tree with root All $l$ are $l$ in which the rule (Axiom) is *not* used.
   (c) Is there a proof tree with root All $n$ are $p$ whose leaves are all in $\Gamma$ and which uses (Barbara) at least once? If so, draw it.

5. Suppose $\Gamma \vdash \varphi$ in $\mathsf{A}$. True or false, with a reason:
   (a) There are infinitely many proof trees over $\Gamma$ with root $\varphi$.
   (b) Any two proof trees over $\Gamma$ with root $\varphi$ must share a leaf taken from $\Gamma$.

(c) If a sentence of the form All $p$ are $p$ occurs in a proof tree, that occurrence must be at a leaf.

6. Let $\mathsf{P} = \{x_1, x_2, \dots\}$ and

$$\Gamma = \{\text{All } x_1 \text{ are } x_2,\ \text{All } x_2 \text{ are } x_3,\ \dots,\ \text{All } x_n \text{ are } x_{n+1},\ \dots\}.$$

(a) Use induction on proof trees over $\Gamma$ to show: if $\Gamma \vdash$ All $x_i$ are $x_j$ then $i \le j$.

(b) Exhibit a model of $\Gamma$ with $[\![x_1]\!] \subset [\![x_2]\!] \subset [\![x_3]\!] \subset \cdots$ (proper inclusions) and use soundness or completeness to get the same result as in (a).

7. Let $\Gamma \subseteq \mathcal{A}$. Prove that for all nouns $p, q$: $p \to_\Gamma^* q$ iff $p \le_\Gamma q$. [One direction is an induction on the length of a path; the other is an induction on proof trees.]

8. Let $\Gamma = \{\text{All } x \text{ are } y,\ \text{All } z \text{ are } y,\ \text{All } y \text{ are } w\}$ and $\varphi =$ All $x$ are $z$. True or false, with a reason, using the original definitions as much as possible; whenever you use soundness or completeness, say so.

(a) $\Gamma \models \varphi$.

(b) $\Gamma \vdash \varphi$.

(c) $x \le_\Gamma w$.

(d) $x \to_\Gamma w$.

(e) $x \to_\Gamma^* w$.

9. Let

$$\Gamma = \left\{ \begin{array}{lll} \text{All } j \text{ are } k, & \text{All } j \text{ are } l, & \text{All } k \text{ are } l, \\ \text{All } l \text{ are } k, & \text{All } l \text{ are } m, & \text{All } k \text{ are } n, \\ \text{All } m \text{ are } q, & \text{All } p \text{ are } q, & \text{All } q \text{ are } p \end{array} \right\}.$$

(a) Draw the all-graph of $\Gamma$ on $S = \{j, k, l, m, n, p, q\}$ and the associated preorder (Hasse diagram, with equivalent nouns placed together).

(b) Decide, using the algorithm: $\Gamma \models$ All $p$ are $n$? $\Gamma \models$ All $j$ are $n$? $\Gamma \models$ All $m$ are $p$? $\Gamma \models$ All $n$ are $m$?

(c) For each "no" in (b) write down the counter-model produced by the algorithm ($N = S$, $[\![u]\!] = \{v : v \to_\Gamma^* u\}$) and check the sentence fails in it. For each "yes" turn the path into a proof tree.

10. Let $\Gamma \subseteq \mathcal{A}$ be arbitrary and fix nouns $p, q$ with $\Gamma \models$ All $p$ are $q$. Let $N = \{*\}$ and $[\![u]\!] = \{*\}$ if $\Gamma \vdash$ All $p$ are $u$, $[\![u]\!] = \varnothing$ otherwise.

(a) Show $\mathcal{N} \models \Gamma$.

(b) Show $\mathcal{N} \models$ All $p$ are $q$.

(c) Use (b) to show $\Gamma \vdash$ All $p$ are $q$.

(d) Explain in two sentences why (a)–(c) constitute a second proof of the completeness theorem.

11. In the canonical model we took the universe to be $\mathsf{P}$, the set of *all* nouns. Suppose instead we used only the nouns occurring in $\Gamma$. Would the completeness proof still work? What, if anything, has to change? [Consider $\varphi =$ All $p$ are $q$ where $p$ does not occur in $\Gamma$.]

12. Let $\Gamma$ be a set of sentences in $\mathcal{A}$ (or in any logic extending $\mathcal{A}$), and let $\mathcal{M}$ be a model whose interpretation function is monotone with respect to $\leq_\Gamma$: if $p \leq_\Gamma q$ then $[\![p]\!] \subseteq [\![q]\!]$. Prove that $\mathcal{M} \models \Gamma_{\text{all}}$.

13. A model $\mathcal{M}$ is a *characteristic model* of $\Gamma$ if for every sentence $\varphi$ of the language, $\mathcal{M} \models \varphi$ iff $\Gamma \vdash \varphi$.
    (a) Explain why the canonical model of $\Gamma$ in $\mathcal{A}$ is a characteristic model of $\Gamma$.
    (b) Show that in the logic $\mathcal{S}$ (with the proof system $\mathsf{S}$) the empty set $\Gamma = \varnothing$ has *no* characteristic model. [Consider Some $p$ are $p$ and No $p$ are $p$.]
    (c) Does $\varnothing$ have a characteristic model in $\mathcal{S}^\dagger$?

14. Prove directly from the definitions:
    (a) All $p$ are $q$, Some $p$ are $r \models$ Some $q$ are $r$.
    (b) All $p$ are $v$, All $q$ are $w$, No $v$ are $w \models$ No $p$ are $q$.
    (c) Some $p$ are $q$, Some $q$ are $n$, All $q$ are $m \not\models$ Some $p$ are $n$. [Use the two *Some* sentences themselves as the points of the model.]

15. Give derivations in $\mathsf{S}$ of the following. Label every rule.
    (a) All $n$ are $q$, All $n$ are $p$, Some $n$ are $n \vdash$ Some $p$ are $q$.
    (b) All $p$ are $v$, All $q$ are $w$, No $v$ are $w \vdash$ No $p$ are $q$.
    (c) No $p$ are $q \vdash$ No $q$ are $p$ (the derived rule (Anti)).
    (d) Some $p$ are $q$, All $q$ are $r \vdash$ Some $r$ are $p$.
    (e) All $p$ are $q$, No $r$ are $q \vdash$ No $p$ are $r$.

16. Let $\Gamma = \{$All $a$ are $b$, All $b$ are $c$, No $c$ are $a$, Some $a$ are $a\}$.
    (a) Show that $\Gamma \vdash$ Some $a$ are $c$ and $\Gamma \vdash$ No $a$ are $c$.
    (b) Conclude that $\Gamma \vdash \psi$ for every sentence $\psi$ of $\mathcal{S}$. Which rule does this?
    (c) Give a semantic explanation of (b): what do the models of $\Gamma$ look like?

17. In class we built a canonical model only for $\mathcal{A}$. This question introduces one for $\mathcal{S}$; it is called $\mathcal{M}_2$. Let $\Gamma$ be a set of sentences of $\mathcal{S}$. The universe of $\mathcal{M}_2$ is $\Gamma_{\text{some}}$, the set of *Some*-sentences in $\Gamma$: the points of the model are sentences. For a noun $u$,

$$[\![u]\!] = \{\text{Some } x \text{ are } y \in \Gamma : \ \Gamma \vdash \text{All } x \text{ are } u \ \text{ or } \ \Gamma \vdash \text{All } y \text{ are } u\}.$$

    The idea: a sentence Some $x$ are $y$ in $\Gamma$ is a witness that something is both an $x$ and a $y$, and this witness is put into $[\![u]\!]$ exactly when $\Gamma$ proves that such a thing must be a $u$.

    Now let $\Gamma = \{$Some $a$ are $b$, Some $c$ are $d$, All $b$ are $d$, All $a$ are $e\}$.
    (a) Write down $[\![a]\!], [\![b]\!], [\![c]\!], [\![d]\!], [\![e]\!]$ in $\mathcal{M}_2$.
    (b) Verify directly that $\mathcal{M}_2 \models \Gamma$.
    (c) Find a sentence of the form All $x$ are $y$ that is true in $\mathcal{M}_2$ but not provable from $\Gamma$. Justify "not provable".

(d) Is Some $e$ are $d$ provable from $\Gamma$? Is Some $c$ are $a$? Give a derivation where one exists; where none exists, explain how $\mathcal{M}_2$ and soundness settle the question.

18. Let $\Gamma$ be any set of sentences of $\mathcal{S}$ and let $\mathcal{M}_2$ be the model of Question 17. You may use the following fact without proof:

$$(*) \quad \text{if } \mathcal{M}_2 \models \text{Some } p \text{ are } q \text{ then } \Gamma \vdash \text{Some } p \text{ are } q.$$

(a) Prove that $\mathcal{M}_2 \models \Gamma_{\text{all}}$. Start with a sentence All $x$ are $y$ in $\Gamma$ and a point Some $a$ are $b \in [\![x]\!]$.

(b) Prove that $\mathcal{M}_2 \models \Gamma_{\text{some}}$.

(c) Prove: if $\Gamma \models$ Some $p$ are $q$ then $\Gamma \vdash$ Some $p$ are $q$. Split into two cases: $\mathcal{M}_2 \models \Gamma_{\text{no}}$, and some No $m$ are $n$ in $\Gamma$ is false in $\mathcal{M}_2$. In the second case explain why $\Gamma \vdash$ No $m$ are $n$ and $\Gamma \vdash$ Some $m$ are $n$, and which rule finishes the proof.

(d) Explain why the same argument does *not* prove: if $\Gamma \models$ All $p$ are $q$ then $\Gamma \vdash$ All $p$ are $q$. [Question 19.]

19. In $\mathcal{S}$ let $\Gamma = \{\text{Some } p \text{ are } q\}$ with $p \neq q$, and let $\mathcal{M}_2$ be the model of Question 17.

(a) Show that $\mathcal{M}_2$ satisfies All $p$ are $q$ but $\Gamma \nvdash$ All $p$ are $q$.

(b) Is every *No*-sentence true in $\mathcal{M}_2$ provable from $\Gamma$? Give an example.

20. Let $\Gamma = \{\text{Some } w \text{ are } x,\ \text{Some } w \text{ are } y,\ \text{Some } w \text{ are } z,\ \text{All } z \text{ are } a\}$. True or false, with a good reason; if you use soundness or completeness say which.

(a) $\Gamma \vdash$ Some $a$ are $a$.

(b) $\Gamma \vdash$ Some $x$ are $y$.

21. Let $M = \{1, 2, 3, 4, 5\}$, $[\![p]\!] = \{1, 2\}$, $[\![q]\!] = \{1, 2, 3\}$, $[\![r]\!] = \{1, 3, 4, 5\}$.

(a) Write down $[\![\bar{p}]\!]$, $[\![\bar{q}]\!]$, $[\![\bar{r}]\!]$.

(b) Decide: All $p$ are $q$; All $\bar{q}$ are $r$; Some $\bar{r}$ are $p$; All $\bar{r}$ are $q$; Some $\bar{p}$ are $\bar{q}$; All $p$ are $\bar{r}$; No $p$ are $r$; Some $\bar{q}$ are $\bar{q}$; Some $p$ are $\bar{p}$.

(c) For each point $m \in M$ write down the set $S_m$ of literals $x$ with $m \in [\![x]\!]$.

22. (a) Write the semantic negation $\bar{\varphi}$ of: All $x$ are $y$; Some $x$ are $y$; No $x$ are $y$; Some $x$ are $\bar{y}$; All $\bar{x}$ are $\bar{y}$.

(b) Prove: for every model $\mathcal{M}$ and sentence $\varphi$ of $\mathcal{S}^\dagger$, $\mathcal{M} \not\models \varphi$ iff $\mathcal{M} \models \bar{\varphi}$.

(c) Show that $\bar{\bar{\varphi}} = \varphi$.

(d) Translate into $\mathcal{S}^\dagger$: "Some farmers are not artists"; "No non-farmer is an artist"; "Every non-artist is a non-farmer". Which of these are equivalent to All $f$ are $a$?

23. True or false, with a reason:

(a) For every model $\mathcal{M}$ and sentence $\varphi$ of $\mathcal{S}^\dagger$: $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \bar{\varphi}$.

(b) For every set $\Gamma$ and sentence $\varphi$ of $\mathcal{S}^\dagger$: $\Gamma \models \varphi$ or $\Gamma \models \bar{\varphi}$.

(c) Every set of sentences in $\mathcal{S}^\dagger$ has a model.

(d) Every set of *All*-sentences in $\mathcal{S}^\dagger$ has a model.

(e) If $\Gamma \models$ Some $p$ are $q$ then $\Gamma_{\text{some}} \neq \varnothing$.

24. This problem is about $\mathsf{S}^\dagger$. Let $\Gamma = \{\text{All } q \text{ are } m,\ \text{All } \bar{n} \text{ are } \bar{m},\ \text{All } q \text{ are } \bar{n}\}$.
   (a) Use $\mathsf{S}^\dagger$ to show All $\bar{n}$ are $\bar{m} \vdash$ All $m$ are $n$. [Hint: (RAA).]
   (b) Show $\Gamma \vdash$ All $q$ are $n$. You may use (a).
   (c) Find a model of $\Gamma$ in which All $n$ are $m$ is false.
25. Show the following in $\mathsf{S}^\dagger$.
   (a) All $p$ are $\bar{q}$ $\vdash$ All $q$ are $\bar{p}$ (E-conversion), and more generally All $x$ are $y$ $\vdash$ All $\bar{y}$ are $\bar{x}$ (contraposition).
   (b) Some $p$ are $q$, No $q$ are $n \vdash$ Some $p$ are $\bar{n}$ (Ferio).
   (c) All $q$ are $n$, All $q$ are $\bar{n} \vdash$ No $q$ are $q$ (complement inconsistency).
   (d) All $y$ are $p$, All $\bar{y}$ are $p$, All $\bar{q}$ are $\bar{z}$, Some $x$ are $z \vdash$ Some $p$ are $q$.
26. Let $\Gamma = \{\text{All } y \text{ are } p,\ \text{All } p \text{ are } q,\ \text{All } q \text{ are } y,\ \text{All } \bar{y} \text{ are } p,\ \text{All } q \text{ are } z\}$.
   (a) Show semantically that $\Gamma \models$ All $x$ are $z$ for every noun $x$ (including nouns not in $\Gamma$).
   (b) Give a derivation in $\mathsf{S}^\dagger$ of All $x$ are $z$ from $\Gamma$. [You will need (RAA) once, to get All $x$ are $p$ from All $y$ are $p$ and All $\bar{y}$ are $p$; compare the rule (One).]
27. (a) Prove: if $\Gamma \cup \{\varphi\} \vdash \bot$ then $\Gamma \vdash \bar{\varphi}$.
   (b) State the rule $(\chi)$ of $\mathsf{S}$ in the form "if ... then ..." about derivability, and do the same for (RAA). What is the essential difference between the two?
   (c) Show that $(\chi)$ is a derived rule of $\mathsf{S}^\dagger$: if $\Gamma \vdash$ Some $p$ are $q$ and $\Gamma \vdash$ All $p$ are $\bar{q}$ then $\Gamma \vdash \psi$ for every $\psi$.
28. The *Chinese lantern* $\mathbf{P}_2$ has six points $0 < x_1, x_2, x_3, x_4 < 1$ (the four $x_i$ pairwise incomparable) with $\bar{0} = 1$, $\bar{x_1} = x_2$, $\bar{x_3} = x_4$.
   (a) Verify that $\mathbf{P}_2$ is an orthoposet.
   (b) Find all states of $\mathbf{P}_2$.
   (c) Which of the following are states: $\{1\}$, $\{x_1, 1\}$, $\{x_1, x_2, 1\}$, $\{x_1, x_3, x_4, 1\}$? Say which condition fails.
29. Let $\mathbf{P} = (P, \leq, 0, ^-)$ be an orthoposet.
   (a) Let $q$ be such that $p \leq q$ for all $p$. Show $\bar{q} = 0$.
   (b) Show that $S \subseteq P$ is a state iff $S$ is up-closed and for each $p$ exactly one of $p, \bar{p}$ belongs to $S$.
30. Let $\mathbf{P}$ be an orthoposet with $0 \neq 1$ and let $S$ be a state of $\mathbf{P}$.
   (a) Show that $1 \in S$ and $0 \notin S$.
   (b) Show that if $p \in S$ and $q \notin S$ then $p \not\leq q$.
   (c) Show that $S$ is extendible.
   (d) Show that if $p \leq \bar{q}$ then $p$ and $q$ are never both in a state. Conclude that if $p \leq \bar{p}$ (that is, $p = 0$) then $p$ belongs to no state.

31. Let $\mathbf{P}$ be an orthoposet.
    (a) Prove that every state is extendible.
    (b) Prove that $\{p, q\}$ is not extendible iff $p \leq \bar{q}$.
    (c) Prove: if $S$ is extendible then $S \cup \{p\}$ or $S \cup \{\bar{p}\}$ is extendible. Write out at least two of the cases.
    (d) Prove: $S_0$ is contained in a state iff $S_0$ is extendible (finite $P$). Where in the proof is up-closure checked?

32. Let

$$\Gamma = \{\text{All } y \text{ are } x,\ \text{All } \bar{y} \text{ are } x,\ \text{All } z \text{ are } y,\ \text{All } \bar{z} \text{ are } \bar{y},\ \text{All } \bar{z} \text{ are } w\}.$$

    (a) Compute the equivalence classes of literals under $\equiv_\Gamma$ and draw the Hasse diagram of $\mathbf{P}_\Gamma$. Which class is 0, which is 1?
    (b) List all states of $\mathbf{P}_\Gamma$.
    (c) Write down the third canonical model $\mathcal{M}_3$: its universe and $[\![x]\!], [\![y]\!], [\![z]\!], [\![w]\!]$.
    (d) Use $\mathcal{M}_3$ to show that $\Gamma \nvdash$ All $y$ are $w$. Which theorem are you using?

33. Let $\Gamma = \{\text{All } a \text{ are } c,\ \text{All } b \text{ are } c,\ \text{All } c \text{ are } d,\ \text{All } c \text{ are } e\}$.
    (a) Draw $\mathbf{P}_\Gamma$. (All classes are singletons; a fresh 0 and 1 must be added. Why?)
    (b) How many states does $\mathbf{P}_\Gamma$ have? List them.
    (c) Build a state containing $\{a, \bar{b}\}$ with the algorithm from class (go through the remaining literals one at a time, adding a literal if the set stays extendible and its complement otherwise), processing the literals in the order $c, d, e, \bar{a}, \ldots$.
    (d) In $\mathcal{M}_3$, decide: All $a$ are $d$? All $d$ are $b$? Some $a$ are $b$? Explain each answer with the states.

34. (a) Prove that if $\Gamma$ is consistent then $\mathcal{M}_3 \models \Gamma$. Treat the *All* case and the *Some* case separately; the *Some* case must use consistency.
    (b) Deduce the completeness theorem for $\mathsf{S}^\dagger$ from (a) and the fact that $\Gamma \cup \{\varphi\} \vdash \bot$ implies $\Gamma \vdash \bar{\varphi}$.
    (c) Deduce that $\Gamma$ is consistent iff $\Gamma$ is satisfiable.

35. Let $\Gamma \subseteq \mathcal{S}^\dagger$ be consistent and let $n$ be the number of nouns. Show that $\Gamma$ has a model with at most $2^n$ elements.

36. For each set, decide whether it is consistent in $\mathsf{S}^\dagger$. Give a derivation of $\bot$ or a model.
    (a) $\{\text{All } a \text{ are } b,\ \text{Some } a \text{ are } \bar{b}\}$
    (b) $\{\text{All } a \text{ are } b,\ \text{All } \bar{b} \text{ are } a\}$
    (c) $\{\text{Some } a \text{ are } a,\ \text{All } a \text{ are } b,\ \text{All } a \text{ are } \bar{b}\}$
    (d) $\{\text{All } p \text{ are } q,\ \text{All } q \text{ are } r,\ \text{Some } r \text{ are } p\}$
    (e) $\{\text{All } p \text{ are } \bar{p},\ \text{Some } q \text{ are } q\}$
    (f) $\{\text{All } p \text{ are } \bar{p},\ \text{All } \bar{p} \text{ are } p\}$

37. True or false, with a reason (all in $\mathsf{S}^\dagger$):
    (a) If $\Gamma_{\text{all}}$ is consistent then $\Gamma$ is consistent.
    (b) If $\Gamma$ is consistent and $\Gamma \vdash$ All $p$ are $q$, then $\Gamma_{\text{all}} \vdash$ All $p$ are $q$.
    (c) If $\Gamma$ is inconsistent then $\Gamma \vdash$ All $p$ are $q$ for all $p, q$.
    (d) If $\Gamma \vdash$ Some $p$ are $q$ and $\Gamma$ is consistent, then some sentence of $\Gamma$ is a *Some*-sentence.
    (e) If $\Gamma \models$ Some $p$ are $p$ for every noun $p$, then $\Gamma$ is inconsistent.

38. True or false, with a reason:
    (a) If a logical system is sound and we add more rules to it, the larger system is automatically sound.
    (b) If a logical system is complete and we add more rules to it, the larger system is automatically complete.
    (c) Every set of sentences in $\mathcal{S}^\dagger$ has a model.
    (d) In $\mathcal{S}^\dagger$: for all $\Gamma$ and $\varphi$, either $\Gamma \models \varphi$ or $\Gamma \models \bar{\varphi}$.

39. Answer each in one or two sentences.
    (a) Suppose a proof system for $\mathcal{A}$ has *no* rules at all (so $\Gamma \vdash \varphi$ iff $\varphi \in \Gamma$). Is it sound? Is it complete?
    (b) Suppose a proof system for $\mathcal{A}$ has the single rule "from any premises, infer any sentence". Is it sound? Is it complete?
    (c) In $\mathsf{A}$, is $\Gamma \vdash \varphi$ decidable, i.e. is there an algorithm that always answers? Name it.
    (d) Give an example of a set $\Gamma$ in $\mathcal{A}$ and a sentence $\varphi$ such that $\Gamma \vdash \varphi$ has a proof tree with 3 leaves from $\Gamma$ but none with fewer.

# Definitions

## Models and truth

A **model** is a set $M$ (the universe, possibly empty) together with a subset $[\![p]\!] \subseteq M$ for each noun $p$. In $\mathcal{S}^\dagger$ the complemented noun $\bar{p}$ is interpreted as $[\![\bar{p}]\!] = M \setminus [\![p]\!]$; *literal* means a noun or a complemented noun.

$$\begin{array}{lll} \mathcal{M} \models \text{All } p \text{ are } q & \text{iff} & [\![p]\!] \subseteq [\![q]\!] \\ \mathcal{M} \models \text{Some } p \text{ are } q & \text{iff} & [\![p]\!] \cap [\![q]\!] \neq \varnothing \\ \mathcal{M} \models \text{No } p \text{ are } q & \text{iff} & [\![p]\!] \cap [\![q]\!] = \varnothing \qquad (\text{in } \mathcal{S}^\dagger\text{: No } p \text{ are } q \text{ is All } p \text{ are } \bar{q}) \end{array}$$

$\mathcal{M} \models \Gamma$ means $\mathcal{M} \models \varphi$ for every $\varphi \in \Gamma$.

**Parts of $\Gamma$:** $\Gamma_{\text{all}} = \{\varphi \in \Gamma : \varphi \text{ is of the form All } p \text{ are } q\}$, and likewise $\Gamma_{\text{some}}$ and $\Gamma_{\text{no}}$; $\Gamma_{\text{all,some}} = \Gamma_{\text{all}} \cup \Gamma_{\text{some}}$. In $\mathcal{S}^\dagger$ every *No*-sentence is an *All*-sentence, so $\Gamma = \Gamma_{\text{all}} \cup \Gamma_{\text{some}}$.

**Semantic negation** (in $\mathcal{S}^\dagger$): $\overline{\text{All } x \text{ are } y} = \text{Some } x \text{ are } \bar{y}$, $\overline{\text{Some } x \text{ are } y} = \text{All } x \text{ are } \bar{y}$.

## Consequence, derivability, soundness, completeness

- $\Gamma \models \varphi$ (**semantic consequence**): every model of $\Gamma$ is a model of $\varphi$.
- $\Gamma \vdash \varphi$ (**derivability**): there is a proof tree with root $\varphi$ whose leaves are sentences of $\Gamma$ or instances of (Axiom); in $\mathsf{S}^\dagger$, leaves withdrawn by (RAA) are also allowed. A **proof tree** is built by applying the rules of the system; a one-point tree labelled by $\varphi \in \Gamma$ shows $\Gamma \vdash \varphi$.
- A proof system is **sound** if for all $\Gamma, \varphi$: if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.
- A proof system is **complete** if for all $\Gamma, \varphi$: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.
- $\perp$ stands for any sentence Some $p$ are $\bar{p}$. $\Gamma$ is **consistent** if $\Gamma \nvdash \perp$ (equivalently, $\Gamma$ does not derive every sentence); otherwise **inconsistent**.
- $\Gamma$ is **satisfiable** if it has a model.

## Orders, the all-graph, the star

- A **preorder** is a reflexive and transitive relation. $p \leq_\Gamma q$ iff $\Gamma \vdash$ All $p$ are $q$. $p \equiv_\Gamma q$ iff $p \leq_\Gamma q$ and $q \leq_\Gamma p$.
- **All-graph** $\to_\Gamma$: the nouns are the points, and $p \to_\Gamma q$ iff the sentence All $p$ are $q$ belongs to $\Gamma$ (membership, not derivability).
- $p \to_\Gamma^* q$: there is a path $p = p_0 \to p_1 \to \cdots \to p_n = q$ with $n \geq 0$ (reflexive-transitive closure). So $p \to_\Gamma^* p$ always.
- In $\mathcal{S}^\dagger$ the graph $\to_\Gamma$ is on literals: All $p$ are $q \in \Gamma$ gives the two edges $p \to q$ and $\bar{q} \to \bar{p}$. A literal $p$ is a **zero-point** if $p \to^* \bar{p}$, a **one-point** if $\bar{p} \to^* p$; $p \preceq q$ iff $p \to^* q$ or $p$ is a zero-point or $q$ is a one-point.

## Canonical models

- $\mathcal{M}$ for $\mathcal{A}$: universe $\mathsf{P}$ (all nouns), $[\![u]\!] = \{v : \Gamma \vdash \text{All } v \text{ are } u\} = {\downarrow} u$.
- $\mathcal{M}_3$ for $\mathcal{S}^\dagger$: universe = the set of states of $\mathbf{P}_\Gamma$, and $[\![u]\!] = \{S : [u] \in S\}$.

## Orthoposets and states

- A **poset** is a set with a reflexive, transitive, antisymmetric relation $\leq$. An **orthoposet** $(P, \leq, 0, ^{-})$ is a poset with a least element 0 and a map $p \mapsto \bar{p}$ such that $\bar{\bar{p}} = p$; $p \leq q$ implies $\bar{q} \leq \bar{p}$; and (complement inconsistency) $p \leq q$ and $p \leq \bar{q}$ imply $p = 0$. Write $1 = \bar{0}$ (the largest element).
- $\mathbf{P}_{\Gamma}$: the literals modulo $\equiv_{\Gamma}$, with $[p] \leq [q]$ iff $\Gamma \vdash$ All $p$ are $q$, $\bar{[p]} = [\bar{p}]$, and $0 = [p]$ for any $p$ with $\Gamma \vdash$ All $p$ are $\bar{p}$ (if there is none, a fresh 0 and 1 are added).
- A **state** $S \subseteq P$ is up-closed ($p \in S$, $p \leq q$ imply $q \in S$), complete ($p \in S$ or $\bar{p} \in S$ for every $p$), consistent (never both $p$ and $\bar{p}$), and non-empty. Equivalently: up-closed and containing exactly one of $p, \bar{p}$ for each $p$.
- $S_0 \subseteq P$ is **extendible** if there are no $p, q \in S_0$ with $p \leq \bar{q}$.

# Proof rules

$$\dfrac{}{\text{All } p \text{ are } p}\ \text{Axiom} \qquad \dfrac{\text{All } p \text{ are } n \qquad \text{All } n \text{ are } q}{\text{All } p \text{ are } q}\ \text{Barbara}$$

The logical system $\mathsf{A}$ for $\mathcal{A}$.

$$\dfrac{}{\text{All } p \text{ are } p}\ \text{Axiom} \qquad \dfrac{\text{Some } p \text{ are } q}{\text{Some } p \text{ are } p}\ \text{Some}_1 \qquad \dfrac{\text{Some } p \text{ are } q}{\text{Some } q \text{ are } p}\ \text{Some}_2$$

$$\dfrac{\text{All } p \text{ are } n \qquad \text{All } n \text{ are } q}{\text{All } p \text{ are } q}\ \text{Barbara} \qquad \dfrac{\text{All } q \text{ are } n \qquad \text{Some } p \text{ are } q}{\text{Some } p \text{ are } n}\ \text{Darii}$$

$$\dfrac{\text{No } q \text{ are } q}{\text{All } q \text{ are } p}\ \text{Zero} \qquad \dfrac{\text{No } p \text{ are } p}{\text{No } p \text{ are } q}\ \text{No}$$

$$\dfrac{\text{All } p \text{ are } q \qquad \text{No } q \text{ are } r}{\text{No } r \text{ are } p}\ \text{Camestres} \qquad \dfrac{\text{No } p \text{ are } q \qquad \text{Some } p \text{ are } q}{\varphi}\ \chi$$

The logical system $\mathsf{S}$ for $\mathcal{S}$. Derived rule (Anti): from No $p$ are $q$ infer No $q$ are $p$ (Example 3.7).

$$\dfrac{}{\text{All } p \text{ are } p}\ \text{Axiom} \qquad \dfrac{\text{Some } p \text{ are } q}{\text{Some } p \text{ are } p}\ \text{Some}_1 \qquad \dfrac{\text{Some } p \text{ are } q}{\text{Some } q \text{ are } p}\ \text{Some}_2$$

$$\dfrac{\text{All } p \text{ are } n \qquad \text{All } n \text{ are } q}{\text{All } p \text{ are } q}\ \text{Barbara} \qquad \dfrac{\text{All } q \text{ are } n \qquad \text{Some } p \text{ are } q}{\text{Some } p \text{ are } n}\ \text{Darii}$$

$$\dfrac{\dfrac{[\varphi]}{\begin{array}{c}\vdots\\ \hline \bot\end{array}}}{\bar{\varphi}}\ \text{RAA}$$

The logical system $\mathsf{S}^\dagger$ for $\mathcal{S}^\dagger$. All of the literals can be complemented nouns. $\bot$ stands for any sentence Some $p$ are $\bar{p}$; No $p$ are $q$ abbreviates All $p$ are $\bar{q}$. Semantic negation: $\overline{\text{All } x \text{ are } y}$ = Some $x$ are $\bar{y}$, $\overline{\text{Some } x \text{ are } y}$ = All $x$ are $\bar{y}$. Derived rules: (Zero), (One), (No), (Camestres), (Anti), ($\chi$).