# M384: Logic — Question Pool for Exam 2

Weeks 7 to 10: Classical Propositional Logic

1. Dictionary: $m$: The museum is open. $r$: It rains. $w$: We walk to the museum. $t$: We take the tram.
   (a) Translate into propositional logic.
      (i) It rains, but the museum is open.
      (ii) We walk to the museum only if it does not rain.
      (iii) If the museum is open, we walk to the museum unless it rains.
      (iv) We neither walk to the museum nor take the tram, although the museum is open.
      (v) The museum being open is necessary for us to take the tram.
      (vi) If the museum is open, we walk or take the tram, but not both.
   (b) The sentence *It rains and we take the tram or we walk to the museum* can be read in two ways. Give a formula for each reading and describe in one sentence a situation in which the two readings have different truth values.
   (c) Translate into natural English, using the same dictionary: (i) $\neg(r \wedge w)$; (ii) $(m \wedge \neg r) \to (w \vee t)$; (iii) $\neg t \to \neg m$. For (iii), also give a second English sentence with the same meaning that uses "only if".

2. For a formula $\varphi$ let $f(\varphi)$ be the number of occurrences of primitive propositions in $\varphi$, counting repetitions (so $f(p \to p) = 2$), and let $b(\varphi)$ be the number of occurrences of binary connectives in $\varphi$ (so $b(\neg p \to \neg(q \wedge p)) = 2$).
   (a) Give recursive definitions of $f$ and $b$ in the style of Definition 7.14.
   (b) Prove by structural induction that $f(\varphi) = b(\varphi) + 1$ for every formula $\varphi$. Write out every case.
   (c) Is it true that $conn(\varphi) \leq 2 \cdot rank(\varphi)$ for every formula $\varphi$? Prove it by induction or give a counterexample; for a counterexample compute both numbers with Definition 7.14.
   (d) Prove by structural induction that $conn(\varphi) \leq 2^{rank(\varphi)} - 1$ for every formula $\varphi$.

3. (a) Write the complete truth table of $(p \vee (q \wedge r)) \to ((p \vee q) \wedge r)$, with one column for each subformula. Classify the formula (tautology, contradiction or contingent) and describe in one sentence the valuations that make it false.
   (b) For each formula decide whether it is a tautology by searching for a falsifying valuation. If there is one, give every valuation of the primitive propositions that makes the formula false.
      (i) $(p \to q) \to ((p \to \neg q) \to \neg p)$
      (ii) $((p \vee q) \to r) \to ((p \to r) \wedge (q \to r))$
      (iii) $((p \to q) \to r) \to (p \to r)$

4. True or false, with a reason. Here $\varphi, \psi$ are any formulas and $\Gamma$ is any set of formulas.
   (a) If $\varphi \vee \psi$ is a tautology, then $\varphi$ is a tautology or $\psi$ is a tautology.
   (b) If $\varphi \to \psi$ is a contradiction, then $\varphi$ is a tautology and $\psi$ is a contradiction.

(c) If $\varphi$ and $\psi$ are both satisfiable, then $\varphi \wedge \psi$ is satisfiable.

(d) If $\Gamma \models \varphi \to \psi$, then $\Gamma \models \neg\varphi$ or $\Gamma \models \psi$.

(e) If $\Gamma \models \varphi$ and $\Delta \models \psi$, then $\Gamma \cup \Delta \models \varphi \wedge \psi$.

(f) $\varphi \approx \psi$ if and only if $\varphi \models \psi$ and $\psi \models \varphi$.

(g) If $\varphi \approx \psi$, then $\neg\varphi \to \psi$ is a tautology.

(h) Every formula in which $p$ is the only primitive proposition is logically equivalent to one of $p$, $\neg p$, $p \to p$, $\neg(p \to p)$.

5. (a) For each claim decide whether it holds. If it holds, prove it from Definition 8.3 (for example by supposing that a valuation satisfies the left side and makes the right side false). If it fails, give a counterexample and check it.

(i) $p \to (q \wedge r),\ \neg q \models \neg p$

(ii) $p \vee q,\ p \to r,\ q \to r \models r$

(iii) $(p \to q) \to p \models p$

(iv) $p \to q,\ q \to r \models r \to p$

(v) $\neg(p \wedge q),\ q \to r \models p \to \neg r$

(b) Prove directly from Definitions 8.1 and 8.3: $\Gamma \models \varphi \wedge \psi$ if and only if $\Gamma \models \varphi$ and $\Gamma \models \psi$.

(c) Prove directly from the definitions: if $\Gamma \models \varphi \to \psi$ and $\Gamma \models \neg\psi$, then $\Gamma \models \neg\varphi$.

6. (a) For each pair decide whether the two formulas are logically equivalent. For "yes" give a truth table or an argument about an arbitrary valuation; for "no" give a valuation under which the values differ.

(i) $(p \wedge q) \to r$ and $(p \to r) \vee (q \to r)$

(ii) $\neg p \to q$ and $\neg q \to p$

(iii) $p \to (q \to p)$ and $q \to (p \to p)$

(iv) $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$

(v) $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$

(b) Using Table 1 and Proposition 8.13, show by a chain of equivalences that

$$\neg((p \vee q) \to \neg r) \ \approx\ (p \wedge r) \vee (q \wedge r).$$

Name the equivalence you use at each step.

(c) Give a formula of $\mathcal{L}(\neg, \wedge)$ that is equivalent to $p \leftrightarrow q$, and one that is equivalent to $\neg p \to (q \wedge r)$. Justify each with a chain of equivalences or a truth table.

7. (a) Let $\varphi = \neg(p \vee q) \to (r \wedge \neg p)$. Apply the recipe from the proof of Theorem 8.16 to $\varphi$, step by step, and write down the formula $\varphi' \in \mathcal{L}(\neg, \wedge)$ that it produces. Do not simplify.

(b) Let $v_{\mathrm{F}}$ be the valuation with $v_{\mathrm{F}}(p) = \mathrm{F}$ for every primitive proposition $p$. Prove by structural induction that $[\![\varphi]\!]_{v_{\mathrm{F}}} = \mathrm{F}$ for every $\varphi \in \mathcal{L}(\wedge, \vee)$, and conclude that $\{\wedge, \vee\}$ is not adequate.

(c) True or false, with a reason:

(i) If $A \subseteq B$ and $A$ is adequate, then $B$ is adequate.

(ii) Every adequate set of connectives contains $\neg$.

(iii) $\{\neg\}$ together with any one binary connective is adequate.

(iv) If $A$ is adequate, then for every formula $\varphi$ there is exactly one $\varphi' \in \mathcal{L}(A)$ with $\varphi' \approx \varphi$.

8. From here on $\wedge$ and $\vee$ are abbreviations (Definition 9.1).

   (a) Write each formula with $\neg$ and $\to$ only. Do not simplify.

   (i) $(p \wedge q) \vee r$ (ii) $\neg(p \vee \neg q)$ (iii) the instance $(p \wedge q) \to p$ of (L4).

   (b) For each formula decide whether it is an axiom. If it is, name the scheme and the formulas that replace $\varphi, \psi, \chi$. If it is not, explain why no scheme fits.

   (i) $(\neg q \to \neg(p \wedge r)) \to ((p \wedge r) \to q)$

   (ii) $(p \to q) \to (r \to (p \to q))$

   (iii) $(q \vee p) \to (p \vee q)$

   (iv) $(p \wedge (q \to r)) \to (q \to r)$

   (v) $p \to ((p \to q) \to q)$

   (c) Write a deduction for each claim in Fitch notation, using only (assump.), (L1) to (L9) and MP. Write every line; do not use HS, the Deduction Theorem or earlier results.

   (i) $p \to q,\ p \to (q \to r) \vdash p \to r$

   (ii) $q \vdash (p \to q) \wedge q$

   (iii) $\neg r \to \neg(p \wedge q),\ p,\ q \vdash r$

9. (a) Show $\vdash ((p \to q) \wedge (q \to r)) \to (p \to r)$. First say which claim you reduce it to with the Deduction Theorem, then write the deduction in Fitch notation.

   (b) Show $p \vee q \vdash (p \to r) \to ((q \to r) \to r)$ in the same way.

   (c) A student argues: "The Deduction Theorem says that $\Gamma \vdash \varphi \to \psi$ and $\Gamma, \psi \vdash \varphi$ are the same." Correct the statement, and show with $\Gamma = \{\neg p \to q\}$ and the Soundness Theorem that the student's version is false.

10. (a) For each claim, decide whether it holds. If it holds, give a deduction (you may use the Deduction Theorem, HS and results of the notes) or cite the Completeness Theorem with a truth-table argument. If it fails, prove this with the Soundness Theorem, and check your valuation.

   (i) $p \to q \vdash (r \to p) \to (r \to q)$

   (ii) $p \to q \vdash (p \to r) \to (q \to r)$

   (iii) $\neg(p \to q) \vdash p$

   (iv) $\vdash (p \to q) \vee (q \to p)$

   (b) Decide whether each set is consistent, with a reason.

   (i) $\{p \to q,\ q \to r,\ \neg r,\ p \vee r\}$ (ii) $\{p \to q,\ \neg q \to r,\ \neg p,\ r \to \neg q\}$

   (c) Add to (L1) to (L9) the scheme (L10) $(\varphi \vee \psi) \to \varphi$, and write $\vdash'$ for deductions in the new system.

   (i) Show that $\vdash' p$ and $\vdash' \neg p$. Hint: write the instances of (L10) out with Definition 9.1 and use Example 8.27 and Lemma 9.5(c).

(ii) Conclude that $\varnothing$ is inconsistent in the new system.

(iii) Exactly where does the proof of the Soundness Theorem break for the new system?

11. True or false, with a reason.

(a) If $\Gamma \vdash \varphi$ and $\Gamma$ is satisfiable, then $\varphi$ is satisfiable.

(b) If $\nvdash \varphi$, then $\vdash \neg\varphi$.

(c) If $\Gamma \vdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

(d) If we remove (L3) from the list of axioms, the Soundness Theorem still holds for the smaller system.

(e) If we add to (L1) to (L9) a new axiom scheme all of whose instances are tautologies, the Completeness Theorem still holds for the larger system.

(f) $\varphi$ is a theorem if and only if $\{\neg\varphi\}$ is inconsistent.

(g) If $\Gamma$ is consistent, then for every formula $\varphi$ exactly one of $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ holds.

(h) If $\Gamma \models \varphi$, then there is a deduction of $\varphi$ from $\Gamma$ that uses only finitely many members of $\Gamma$.

12. (a) Let $\Gamma$ be maximally consistent with $p \notin \Gamma$, $q \in \Gamma$ and $\neg r \in \Gamma$. For each formula decide whether it belongs to $\Gamma$. Give the reason: which part of Proposition 10.10 you use and what it says about the parts of the formula.

(i) $p \to r$ (ii) $q \to r$ (iii) $\neg(q \wedge \neg r)$ (iv) $(p \vee r) \to q$ (v) $\neg p \to \neg q$ (vi) $(r \to p) \to (p \to q)$ (vii) $\neg(p \vee r) \wedge (q \vee r)$

(b) Let $\Gamma$ and $\Delta$ be maximally consistent with $\Gamma \subseteq \Delta$. Prove that $\Gamma = \Delta$.

(c) True or false, with a reason:

(i) The union of two maximally consistent sets is maximally consistent.

(ii) Every maximally consistent set contains every tautology.

(iii) There is a maximally consistent set that contains $p \to q$ and $\neg q$ but not $\neg p$.

(iv) The set $\{\varphi : \vdash \varphi\}$ of all theorems is maximally consistent.

(v) If $\Gamma$ is maximally consistent and $p \in \Gamma$, then $\Gamma \setminus \{p\}$ is consistent but not maximally consistent.

13. Let $\Gamma = \{\neg q \to p\}$.

(a) Suppose the list of all formulas begins $\varphi_1 = \neg p$, $\varphi_2 = q \to p$, $\varphi_3 = \neg q$, $\varphi_4 = p \to q$, $\varphi_5 = q$. Carry out the first five stages of the construction in the proof of Lemma 10.13. For each stage say whether $\varphi_n$ is added or skipped, write $\Gamma_n$, and justify the decision: a satisfying valuation if the set is consistent, and an argument that no valuation satisfies it (Corollary 10.22) if it is not.

(b) Now suppose the list begins $\varphi_1 = p$, $\varphi_2 = \neg p$, $\varphi_3 = \neg q$. Compute $\Gamma_1, \Gamma_2, \Gamma_3$ with justification.

(c) For the run of part (a), determine $v_{\Gamma'}(p)$ and $v_{\Gamma'}(q)$. After which stage are both values already fixed, whatever the rest of the list is? Explain.

(d) What do parts (a) and (b) show about the maximally consistent set $\Gamma'$ that Lemma 10.13 produces?

(e) For the run of part (a), decide with the Truth Lemma whether $(q \to p) \to \neg q$ and $\neg p \to (q \to p)$ belong to $\Gamma'$.

14. (a) Decide each claim. If it holds, give a truth table or an argument about an arbitrary valuation that satisfies the left side, and name the theorem that turns this into a proof of the claim. If it fails, give a valuation that shows this.

(i) $p \to q,\ q \to r,\ \neg r \vdash \neg p$ (ii) $p \vee q,\ p \to r \vdash r$ (iii) $\neg p \to q,\ \neg p \to \neg q \vdash p$

(b) Let $\Gamma = \{\neg p_1\} \cup \{p_n \to p_{n+1} :\ n \geq 1\} \cup \{p_5\}$. Show that $\Gamma$ is consistent, that $\Gamma \vdash p_{100}$, and that $\Gamma \nvdash p_3$.

(c) Let $\Delta = \{p_n \vee p_{n+1} :\ n \geq 1\} \cup \{\neg p_n :\ n \geq 1\}$. Show that $\Delta$ is unsatisfiable. Which theorem guarantees that some finite subset of $\Delta$ is unsatisfiable? Give such a subset with as few members as possible, and explain why no smaller subset works.

(d) True or false, with a reason:

(i) If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.

(ii) If $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \cdots$ are all satisfiable, then $\bigcup_n \Gamma_n$ is satisfiable.

(iii) Every unsatisfiable set has an unsatisfiable subset with at most two members.

(iv) If $\Gamma$ is infinite and $\Gamma \models \varphi$, then $\Gamma_0 \models \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.

# Reference sheet

## Axioms and rules

An axiom is any instance of one of the following schemes ($\varphi, \psi, \chi$ any formulas).

| | | |
|---|---|---|
| Implication | (L1) | $\varphi \to (\psi \to \varphi)$ |
| | (L2) | $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ |
| Negation | (L3) | $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ |
| Conjunction | (L4) | $(\varphi \wedge \psi) \to \varphi$ |
| | (L5) | $(\varphi \wedge \psi) \to \psi$ |
| | (L6) | $\varphi \to (\psi \to (\varphi \wedge \psi))$ |
| Disjunction | (L7) | $\varphi \to (\varphi \vee \psi)$ |
| | (L8) | $\psi \to (\varphi \vee \psi)$ |
| | (L9) | $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$ |

Rule of inference MP: from $\varphi$ and $\varphi \to \psi$ infer $\psi$. Derived rule HS (Proposition 8.36): from $\varphi \to \psi$ and $\psi \to \chi$ infer $\varphi \to \chi$. Since Week 9, $\varphi \wedge \psi$ abbreviates $\neg(\varphi \to \neg\psi)$ and $\varphi \vee \psi$ abbreviates $\neg\varphi \to \psi$ (Definition 9.1); $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

## Table 1: standard equivalences

| | |
|---|---|
| Double negation | $\neg\neg\varphi \approx \varphi$ |
| De Morgan | $\neg(\varphi \wedge \psi) \approx \neg\varphi \vee \neg\psi, \quad \neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$ |
| Implication | $\varphi \to \psi \approx \neg\varphi \vee \psi, \quad \varphi \to \psi \approx \neg(\varphi \wedge \neg\psi)$ |
| Negated implication | $\neg(\varphi \to \psi) \approx \varphi \wedge \neg\psi$ |
| Contraposition | $\varphi \to \psi \approx \neg\psi \to \neg\varphi$ |
| Exportation | $(\varphi \wedge \psi) \to \chi \approx \varphi \to (\psi \to \chi)$ |
| Commutativity | $\varphi \wedge \psi \approx \psi \wedge \varphi, \quad \varphi \vee \psi \approx \psi \vee \varphi$ |
| Associativity | $(\varphi \wedge \psi) \wedge \chi \approx \varphi \wedge (\psi \wedge \chi), \quad (\varphi \vee \psi) \vee \chi \approx \varphi \vee (\psi \vee \chi)$ |
| Distributivity | $\varphi \wedge (\psi \vee \chi) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ |
| | $\varphi \vee (\psi \wedge \chi) \approx (\varphi \vee \psi) \wedge (\varphi \vee \chi)$ |

## Definitions

- Convention 7.5: the outermost pair of parentheses may be dropped, and $\neg$ applies to the smallest formula after it ($\neg p \vee q$ means $(\neg p) \vee q$). No other parentheses may be dropped.
- Def. 7.14, with $\bullet$ any binary connective:

$$\begin{array}{lll} \mathit{length}(p) = 1, & \mathit{length}(\neg\varphi) = \mathit{length}(\varphi) + 1, & \mathit{length}(\varphi \bullet \psi) = \mathit{length}(\varphi) + \mathit{length}(\psi) + 1, \\ \mathit{conn}(p) = 0, & \mathit{conn}(\neg\varphi) = \mathit{conn}(\varphi) + 1, & \mathit{conn}(\varphi \bullet \psi) = \mathit{conn}(\varphi) + \mathit{conn}(\psi) + 1, \\ \mathit{rank}(p) = 0, & \mathit{rank}(\neg\varphi) = \mathit{rank}(\varphi) + 1, & \mathit{rank}(\varphi \bullet \psi) = \max\{\mathit{rank}(\varphi), \mathit{rank}(\psi)\} + 1. \end{array}$$

  Def. 7.15: $\mathit{sub}(\varphi)$ is the set of subformulas of $\varphi$; $\mathit{occ}(\varphi)$ is the set of primitive propositions occurring in $\varphi$.
- Def. 7.21, 7.22: a **valuation** is a function $v : \mathrm{PROP} \to \{\mathrm{T}, \mathrm{F}\}$; $[\![\varphi]\!]_v$ is defined by recursion with the truth tables of the connectives.

- Def. 7.30. For a formula $\varphi$:

  $\varphi$ is a **tautology** ($\models \varphi$) if $[\![\varphi]\!]_v = \mathrm{T}$ for every valuation $v$;
  $\varphi$ is a **contradiction** if $[\![\varphi]\!]_v = \mathrm{F}$ for every valuation $v$;
  $\varphi$ is **satisfiable** if $[\![\varphi]\!]_v = \mathrm{T}$ for some valuation $v$;
  $\varphi$ is **contingent** if it is neither a tautology nor a contradiction.

- Def. 8.1: $v$ **satisfies** $\Gamma$ if $[\![\psi]\!]_v = \mathrm{T}$ for every $\psi \in \Gamma$; $\Gamma$ is **satisfiable** if some valuation satisfies it.

- Def. 8.3: $\Gamma \models \varphi$ (**semantic consequence**) if every valuation that satisfies $\Gamma$ makes $\varphi$ true. A valuation that satisfies $\Gamma$ and makes $\varphi$ false is a **counterexample**.

- Def. 8.10: $\varphi \approx \psi$ (**logically equivalent**) if $\models \varphi \leftrightarrow \psi$, i.e. $[\![\varphi]\!]_v = [\![\psi]\!]_v$ for every $v$.

- Def. 8.15: a set $A$ of connectives is **adequate** if every formula has an equivalent formula in $\mathcal{L}(A)$.

- Def. 8.26: a **deduction** from $\Gamma$ is a finite sequence of formulas each of which is an axiom, a member of $\Gamma$, or follows from two earlier lines by MP; $\Gamma \vdash \varphi$ if some deduction from $\Gamma$ ends in $\varphi$.

- Def. 9.21: $\Gamma$ is **inconsistent** if $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some $\varphi$; otherwise **consistent**.

- Def. 10.6: $\Gamma$ is **maximally consistent** if it is consistent and every proper superset of $\Gamma$ is inconsistent.

- Def. 10.15: for maximally consistent $\Gamma$, the valuation $v_\Gamma$ has $v_\Gamma(p) = \mathrm{T}$ iff $p \in \Gamma$.

## Main results

- Prop. 7.25: if $v$ and $w$ agree on $occ(\varphi)$, then $[\![\varphi]\!]_v = [\![\varphi]\!]_w$.

- Prop. 7.31:

  (1) $\varphi$ is a tautology iff $\neg\varphi$ is a contradiction;
  (2) $\varphi$ is satisfiable iff $\varphi$ is not a contradiction;
  (3) $\varphi$ is satisfiable iff $\neg\varphi$ is not a tautology.

- Prop. 8.7:

  (a) if $\varphi \in \Gamma$, then $\Gamma \models \varphi$;
  (b) if $\Gamma \subseteq \Delta$ and $\Gamma \models \varphi$, then $\Delta \models \varphi$;
  (c) if $\Gamma \models \varphi$ and $\Gamma \models \varphi \to \psi$, then $\Gamma \models \psi$;
  (d) $\Gamma \cup \{\varphi\} \models \psi$ iff $\Gamma \models \varphi \to \psi$.

- Prop. 8.12: $\approx$ is reflexive, symmetric and transitive.

- Prop. 8.13: if $\varphi \approx \varphi'$ and $\psi \approx \psi'$, then $\neg\varphi \approx \neg\varphi'$, $\varphi \wedge \psi \approx \varphi' \wedge \psi'$, $\varphi \vee \psi \approx \varphi' \vee \psi'$ and $\varphi \to \psi \approx \varphi' \to \psi'$.

- Thm. 8.16: $\{\neg, \wedge\}$ is adequate. The recipe:

  $p' = p, \quad (\neg\varphi)' = \neg\varphi', \quad (\varphi \wedge \psi)' = \varphi' \wedge \psi',$
  $(\varphi \to \psi)' = \neg(\varphi' \wedge \neg\psi'), \quad (\varphi \vee \psi)' = \neg(\neg\varphi' \wedge \neg\psi').$

- Prop. 8.18: $A$ is adequate if there are $N(\varphi) \approx \neg\varphi$ and $K(\varphi, \psi) \approx \varphi \wedge \psi$ in $\mathcal{L}(A)$.

- Example 8.27: $\vdash \varphi \to \varphi$.

- Prop. 8.35:
  (a) if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
  (b) if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$; in particular $\vdash \varphi$ implies $\Gamma \vdash \varphi$;
  (c) if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.
- Prop. 8.36 (HS): if $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$, then $\Gamma \vdash \varphi \to \chi$.
- Thm. 9.7: $\Gamma \vdash_3 \varphi$ iff $\Gamma \vdash_9 \varphi$ (the axioms (L1) to (L3) deduce the same formulas as (L1) to (L9)).
- Lemma 9.4:
  (a) $\vdash (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$;
  (b) if $\Gamma \vdash \varphi \to (\psi \to \chi)$, then $\Gamma \vdash \psi \to (\varphi \to \chi)$;
  (c) $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;
  (d) if $\Gamma \vdash \varphi \to (\varphi \to \psi)$, then $\Gamma \vdash \varphi \to \psi$.
- Lemma 9.5:
  (a) $\vdash \neg\varphi \to (\varphi \to \psi)$;
  (b) $\vdash \varphi \to (\neg\varphi \to \psi)$;
  (c) $\vdash \neg\neg\varphi \to \varphi$;
  (d) $\vdash \varphi \to \neg\neg\varphi$;
  (e) $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$;
  (f) $\vdash (\neg\varphi \to \varphi) \to \varphi$;
  (g) $\vdash (\varphi \to \chi) \to ((\neg\varphi \to \chi) \to \chi)$.
- Thm. 9.9 / Cor. 9.10 (Deduction Theorem): $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \to \psi$.
- Lemma 9.15: every axiom is a tautology. Lemma 9.16: MP preserves truth.
- Thm. 9.17 (Soundness): if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.
- Cor. 9.18: every theorem is a tautology.
- Cor. 9.22: every satisfiable set is consistent; in particular $\varnothing$ is consistent.
- Prop. 10.1:
  (a) $\Gamma$ is inconsistent iff $\Gamma \vdash \psi$ for every formula $\psi$;
  (b) if $\Gamma \subseteq \Delta$ and $\Gamma$ is inconsistent, then $\Delta$ is inconsistent (a subset of a consistent set is consistent).
- Lemma 10.3: $\vdash (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$.
- Prop. 10.4: if $\Gamma \nvdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is consistent.
- Prop. 10.5, for consistent $\Gamma$:
  (a) if $\Gamma \vdash \varphi$, then $\Gamma \cup \{\varphi\}$ is consistent;
  (b) for every $\varphi$, at least one of $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ is consistent.
- Prop. 10.8: $\Gamma$ is maximally consistent iff $\Gamma$ is consistent and, for every $\varphi$, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ (then exactly one).
- Prop. 10.9: if $\Gamma$ is maximally consistent and $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$; every theorem belongs to every maximally consistent set.

- Prop. 10.10 (membership rules), for maximally consistent $\Gamma$:
  (a) $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$;
  (b) $\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$;
  (c) $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$;
  (d) $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.
- Lemma 10.13 (Lindenbaum): every consistent set $\Gamma$ is contained in a maximally consistent set $\Gamma'$. Construction, for a list $\varphi_1, \varphi_2, \ldots$ of all formulas:

  $\Gamma_0 = \Gamma$;
  $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_{n+1}\}$ if this set is consistent, and $\Gamma_{n+1} = \Gamma_n$ otherwise;
  $\Gamma' = \bigcup_{n\geq 0} \Gamma_n$.
- Lemma 10.16 (Truth Lemma): for maximally consistent $\Gamma$ and every $\varphi$, $[\![\varphi]\!]_{v_\Gamma} = \mathrm{T}$ iff $\varphi \in \Gamma$.
- Thm. 10.18: every consistent set is satisfiable.
- Thm. 10.19 (Completeness): if $\models \varphi$, then $\vdash \varphi$.
- Cor. 10.22: $\Gamma$ is consistent iff $\Gamma$ is satisfiable.
- Lemma 10.25: if $\Gamma \vdash \varphi$, then $\Gamma_0 \vdash \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.
- Thm. 10.26 (Compactness): $\Gamma$ is satisfiable iff every finite subset of $\Gamma$ is satisfiable.
- Cor. 10.27: if $\Gamma \models \varphi$, then $\Gamma_0 \models \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.