QUIZ 12 - MATH 225 - SOLUTIONS

1. Consider the first-order system $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $A = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix}$. Determine whether the following sets are bases for the solution space of that system. You should check 1) the vectors are indeed solutions and 2) they are independent.

   (a) $S_1 = \left\{ \begin{bmatrix} e^{5t} \\ e^{5t} \\ 0 \end{bmatrix}, \begin{bmatrix} 4e^{6t} \\ 0 \\ 4e^{6t} \end{bmatrix}, \begin{bmatrix} -e^{2t} \\ 0 \\ e^{-2t} \end{bmatrix} \right\}$ Note that

$$\begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} e^{5t} \\ e^{5t} \\ 0 \end{bmatrix} = \begin{bmatrix} 4e^{5t} \\ 4e^{5t} \\ 0 \end{bmatrix} \neq \begin{bmatrix} e^{5t} \\ e^{5t} \\ 0 \end{bmatrix}'.$$

   Since the vector is not a solution for the given system S_1 cannot be a basis.

   (b) $S_2 = \left\{ \begin{bmatrix} e^{4t} \\ e^{4t} \\ 0 \end{bmatrix}, \begin{bmatrix} e^{6t} \\ 0 \\ e^{6t} \end{bmatrix}, \begin{bmatrix} -e^{6t} \\ e^{6t} \\ 0 \end{bmatrix} \right\}$ We have

$$\begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} e^{4t} \\ e^{4t} \\ 0 \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} e^{6t} \\ 0 \\ e^{6t} \end{bmatrix} = \begin{bmatrix} 6e^{6t} \\ 0 \\ 6e^{6t} \end{bmatrix},$$

$$\begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -e^{6t} \\ e^{6t} \\ 0 \end{bmatrix} = \begin{bmatrix} -6e^{6t} \\ 6e^{6t} \\ 0 \end{bmatrix}.$$

   Thus, all vectors in S_2 are solutions for the given system. We need to check the (in)dependency:

$$W\left(\begin{bmatrix} e^{4t} \\ e^{4t} \\ 0 \end{bmatrix}, \begin{bmatrix} e^{6t} \\ 0 \\ e^{6t} \end{bmatrix}, \begin{bmatrix} -e^{6t} \\ e^{6t} \\ 0 \end{bmatrix} \right)(t) = \det \begin{bmatrix} e^{4t} & e^{6t} & -e^{6t} \\ e^{4t} & 0 & e^{6t} \\ 0 & e^{6t} & 0 \end{bmatrix} = -2e^{16t} \neq 0 \text{ for all } t.$$

   Since they are also independent, S_2 is a basis for the solution space.

   (c) $S_3 = \left\{ \begin{bmatrix} e^{4t} \\ e^{4t} + e^{6t} \\ e^{6t} \end{bmatrix}, \begin{bmatrix} -e^{6t} \\ 2e^{6t} \\ e^{6t} \end{bmatrix}, \begin{bmatrix} e^{4t} + e^{6t} \\ e^{4t} - e^{6t} \\ 0 \end{bmatrix} \right\}$ Observe that these vectors are linear combinations of the vectors in S_2! Since these vectors are the combinations of the solutions, they are also solutions to the same system. So it's enough to check (in)dependency. You can observe that the first vector is the sum of the second and the third. Also, you can use Wronskian:

$$W\left(\begin{bmatrix} e^{4t} \\ e^{4t} + e^{6t} \\ e^{6t} \end{bmatrix}, \begin{bmatrix} -e^{6t} \\ 2e^{6t} \\ e^{6t} \end{bmatrix}, \begin{bmatrix} e^{4t} + e^{6t} \\ e^{4t} - e^{6t} \\ 0 \end{bmatrix} \right)(t) = \det \begin{bmatrix} e^{4t} & -e^{6t} & e^{4t} + e^{6t} \\ e^{4t} + e^{6t} & 2e^{6t} & e^{4t} - e^{6t} \\ e^{6t} & e^{6t} & 0 \end{bmatrix}$$

$$= e^{6t}(-e^{10t} + e^{12t} - 2e^{10t} - 2e^{12t}) - e^{6t}(e^{8t} - e^{10t} - e^{8t} - 2e^{10t} - e^{12t})$$

$$= e^{6t}(-3e^{10t} - e^{12t}) - e^{6t}(3e^{10t} - e^{12t}) = 0$$