QUIZ 5 - MATH 225

Solutions

1. Consider $\mathbb{R}^3$ with usual vector addition and scalar multiplication. Determine whether given subsets are subspaces of $\mathbb{R}^3$ or not.

   - $S_1 = \{(x, y, z) \mid z = x + y\}$
     Let (x, y, z), $(a, b, c) \in S_1$. This means that $z = x + y$ and $c = a + b$. Then
     $$(x, y, z) + (a, b, c) = (x + a, y + b, z + c).$$
     Since $z + c = x + y + a + b = (x + a) + (y + b)$, we have $(x, y, z) + (a, b, c) \in S_1$, so S_1 is closed under addition. Also, if $k \in \mathbb{R}$, then
     $$k(x, y, z) = (kx, ky, kz).$$
     Since $kz = k(x+y) = kx+ky$, we get $k(x, y, z) \in S_1$, so S_1 is closed under scalar multiplication. Thus, S_1 is a subspace of $\mathbb{R}^3$.
   - $S_2 = \{(x, y, z) \mid z = 3\}$
     Since $(0, 0, 0) \notin S_2$, we get S_2 cannot be a subspace of $\mathbb{R}^3$.
   - $S_3 = \{(x, y, z) \mid y = z = 0\}$
     Let (x, y, z), $(a, b, c) \in S_1$. This means that $y = z = 0$ and $b = c = 0$. Then
     $$(x, y, z) + (a, b, c) = (x + a, y + b, z + c).$$
     Since $y + b = z + c = 0 + 0 = 0$, we have $(x, y, z) + (a, b, c) \in S_3$, so S_3 is closed under addition. Also, if $k \in \mathbb{R}$, then
     $$k(x, y, z) = (kx, ky, kz).$$
     Since $ky = kz = k0 = 0$, we get $k(x, y, z) \in S_3$, so S_3 is closed under scalar multiplication. Thus, S_3 is a subspace of $\mathbb{R}^3$.

2. Consider $P_3(\mathbb{R})$, the set of polynomials of degree 3 or less with the usual polynomial addition and scalar multiplication. Determine whether given subset is a subspace of $P_3(\mathbb{R})$ or not.
   $$S = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_1 + a_2 + a_3 = 0\}$$
   Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ be polynomials in S. Then $a_0 + a_1 + a_2 + a_3 = 0$ and $b_0 + b_1 + b_2 + b_3 = 0$. Then
   $$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3.$$
   Since $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0 + 0 = 0$, we get $p(x) + q(x) \in S$, so S is closed under addition. If $k \in \mathbb{R}$, then
   $$kp(x) = (ka_0) + (ka_1)x + (ka_2)x^2 + (ka_3)x^3.$$
   Since $(ka_0) + (ka_1) + (ka_2) + (ka_3) = k0 = 0$, we get $kp(x) \in S$, so S is closed under scalar multiplication. Thus, S is a subspace of $P_2(\mathbb{R})$.