# QUIZ 8 - MATH 225 - Solutions

1. Determine whether the following functions are linear transformations or not.

   (a) $T : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(ax^2 + bx + c) = a + b + c$.
   Suppose $v_1 = a_1x^2 + b_1x + c_1$ and $v_2 = a_2x^2 + b_2x + c_2$. Then, for any scalars $c_1, c_2 \in \mathbb{R}$, we have:

$$
\begin{aligned}
T(c_1v_1 + c_2v_2) &= T(c_1(a_1x^2 + b_1x + c_1) + c_2(a_2x^2 + b_2x + c_2)) \\
&= T((c_1a_1 + c_2a_2)x^2 + (c_1b_1 + c_2b_2)x + (c_1c_1 + c_2c_2)) \\
&= (c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2) + (c_1c_1 + c_2c_2) \\
&= c_1(a_1 + b_1 + c_1) + c_2(a_2 + b_2 + c_2) \\
&= c_1T(v_1) + c_2T(v_2).
\end{aligned}
$$

   (b) $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(a, b) = (a + b, 2)$.

**Long way :)** Let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$. We must have:

$$S(v_1 + v_2) = S((a_1 + a_2, b_1 + b_2)) = ((a_1 + a_2) + (b_1 + b_2), 2).$$

However,

$$S(v_1) + S(v_2) = (a_1 + b_1, 2) + (a_2 + b_2, 2) = (a_1 + b_1 + a_2 + b_2, 4).$$

Since $S(v_1 + v_2) \neq S(v_1) + S(v_2)$, the function $S$ does not satisfy the condition.
Also, for any scalar $\lambda$ and any vector $v = (a, b)$, we should have:

$$S(\lambda v) = S((\lambda a, \lambda b)) = (\lambda a + \lambda b, 2).$$

However,

$$\lambda S(v) = \lambda(a + b, 2) = (\lambda(a + b), 2\lambda).$$

The fact that $S(\lambda v) \neq \lambda S(v)$ shows that $S$ does not satisfy the condition either.

**Short way 1.** $S(0, 0) = (0, 2) \neq (0, 0)$, since $S$ does not map the zero vector to the zero vector, $S$ is not a linear transformation.
**Short way 2.** $2S((1, 1)) = 2(2, 2) = (4, 4)$, but $S(2(1, 1)) = S((2, 2)) = (4, 2)$, since $S$ does not preserve scalar multiplication, $S$ is not a linear transformation.

   (c) $U : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $U(A) = \det(A)$.
   You can use the properties of determinant and show $U$ is not a linear transformation via the long way like before. But it is better to keep it simple and to provide counter examples.
   We have $U(-I_2) = \det(-I_2) = 1$ but $-U(I_2) = -\det(I_2) = -1$. Since $U$ does not preserve scalar product, then $U$ is not a linear transformation.