Linear Algebra & Differential Equations

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10 Week 10

10.1 Kernel and Range

We begin with the definitions.

Definition 10.1. *Let $T : V \to W$ be a linear transformation. The set of vectors of V that is mapped to the zero vector of W by T is called **kernel** of T, denoted as*

$$Ker(T) = \{\mathbf{v} \in V | T(\mathbf{v}) = \mathbf{0}_W\}.$$

*The **range** of the linear transformation T is the subset of W consisting of all transformed vectors from V. We denote the range of T as*

$$Ran(T) = \{T(\mathbf{v}) | \mathbf{v} \in V\}.$$

Examples.

1. We consider the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

   $$T(ax^2 + bx + c) = (2a, 3b).$$

   For the given T, we find the kernel by solving

   $$T(ax^2 + bx + c) = (0, 0),$$

   which gives

   $$(2a, 3b) = (0, 0).$$

   This implies $a = 0$ and $b = 0$, with no restrictions on c. Hence, the kernel of T is composed of all polynomials of the form $0x^2 + 0x + c$, where c is any real number. Therefore,

   $$\text{Ker}(T) = \{c : c \in \mathbb{R}\}.$$

   For $T(ax^2 + bx + c) = (2a, 3b)$, the range includes all ordered pairs in $\mathbb{R}^2$ of the form $(2a, 3b)$, where a and b are any real numbers. Since any vector in $\mathbb{R}^2$ can be written like this, we have

   $$\text{Ran}(T) = \mathbb{R}^2.$$

2. Consider the linear transformation $S : M_2(\mathbb{R}) \to \mathbb{R}^4$ defined by the mapping

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b, c+d, 0, 0).$$

To ascertain the kernel of S, we resolve the equation

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (0, 0, 0, 0),$$

yielding

$$(a+b, c+d, 0, 0) = (0, 0, 0, 0).$$

This necessitates $a + b = 0$ and $c + d = 0$, without imposing any further constraints upon $a, b, c,$ and d. Consequently, the kernel of S is constituted by all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying $a = -b$ and $c = -d$.

$$\text{Ker}(S) = \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} : a, c \in \mathbb{R} \right\}.$$

Given the definition of S, it is evident that the range includes all vectors in $\mathbb{R}^4$ of the form $(a+b, c+d, 0, 0)$, where $a, b, c,$ and d are arbitrary real numbers. Thus,

$$\text{Ran}(S) = \{(x, y, 0, 0) : x, y \in \mathbb{R}\}.$$

3. We examine the linear transformation $U : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$U(a, b) = (a, b, a+b).$$

To find the kernel of U, we set

$$U(a, b) = (0, 0, 0),$$

which leads to the system of equations

$$\begin{aligned} a &= 0, \\ b &= 0, \\ a + b &= 0. \end{aligned}$$

The solution to this system is $a = 0$ and $b = 0$, indicating that the kernel consists only of the zero vector in $\mathbb{R}^2$. Thus,

$$\text{Ker}(U) = \{(0, 0)\}.$$

Given the definition of U, it is apparent that the range includes all vectors in $\mathbb{R}^3$ of the form $(a, b, a+b)$, where a and b are real numbers. Therefore, the range is the entire $\mathbb{R}^3$, as every vector (x, y, z) in $\mathbb{R}^3$ can be written in the form $(a, b, a+b)$ for some $a, b \in \mathbb{R}$ with $x = a$, $y = b$, and $z = a + b$. Hence,

$$\text{Ran}(U) = \{(x, y, z) \in \mathbb{R}^3 : z = x + y\} = span\{(1, 0, 1), (0, 1, 1)\}.$$

The following result is easy but important.

Theorem 10.1. *Let $T; V \to W$ be a linear transformation. Then $Ker(T)$ is a subspace of V, and $Ran(T)$ is a subspace of W.*

Proof. Exercise. □

Remark. Now, consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall that it means that there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. Therefore, we can make following observations:

- $\mathbf{Ker(T) = nullspace(A)}$. The kernel of T, denoted $Ker(T)$, is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ for which $T(\mathbf{x}) = \mathbf{0}$ in $\mathbb{R}^m$, where $\mathbf{0}$ is the zero vector. By the definition of T, we have:
$$T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}.$$
This equation signifies that $\mathbf{x}$ is in the null space of A, which is the set of all vectors that, when multiplied by A, yield the zero vector. Hence,
$$Ker(T) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} = nullspace(A).$$

- $\mathbf{Ran(T) = colspace(A)}$. The range of T, denoted $Ran(T)$, consists of all vectors in $\mathbb{R}^m$ that can be expressed as $T(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^n$. Given $T(\mathbf{x}) = A\mathbf{x}$, every vector in the range of T is a linear combination of the columns of A, since the multiplication of A by $\mathbf{x}$ produces such a linear combination. Therefore, the range of T corresponds to the set of all possible linear combinations of the columns of A, which is precisely the column space of A. Thus,
$$Ran(T) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = colspace(A).$$

Examples.

1. Consider the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix
$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
For any vector $\mathbf{x} = (x, y)$ in $\mathbb{R}^2$, we have
$$T_1(\mathbf{x}) = A_1\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$
Kernel of T_1: The kernel of T_1, $Ker(T_1)$ in this case is
$$Ker(T_1) = \{(x, y) \in \mathbb{R}^2 : x = 0\} = \{(0, y) : y \in \mathbb{R}\},$$
corresponding to the null space of A_1.
Range of T_1: The range of T_1, $Ran(T_1)$, includes all vectors in the form of $(x, 0)$, which forms the x-axis in $\mathbb{R}^2$, and corresponds to the column space of A_1.

2. Consider the linear transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by the matrix

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

The RREF of A_2 is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

This indicates that the first two columns of A_2 are linearly independent and span the column space of A_2, which constitutes the range of T_2.

Therefore, the range (column space) of T_2 is spanned by the vectors

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 5 \end{pmatrix},$$

and can be expressed as

$$Ran(T_2) = \text{Span}\left\{(1,4),(2,5)\right\}.$$

To find the kernel of T_2, we solve the system $A_2\mathbf{x} = \mathbf{0}$, which leads to the following relations:

$$x = z,$$

$$y = -2z.$$

Thus, the kernel of T_2 is spanned by the vector $(1, -2, 1)$, indicating that

$$Ker(T_2) = \{\lambda(1, -2, 1) : \lambda \in \mathbb{R}\}.$$

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 7 \end{pmatrix}$$

and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$.

The RREF of A is

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The kernel of T is given by the solution set to $A\mathbf{x} = \mathbf{0}$, leading to the relations $x = -3z$ and $y = 2z$. Therefore, the kernel of T can be expressed as

$$Ker(T) = \{z(-3, 2, 1) : z \in \mathbb{R}\}.$$

This indicates that the kernel is spanned by the vector $(-3, 2, 1)$, and is a one-dimensional subspace of $\mathbb{R}^3$.

The reduced row echelon form (RREF) of A reveals that the first two columns of A are linearly independent. Thus, the range of T, or the column space of A, is spanned by the vectors

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Therefore, we can express the range as

$$Ran(T) = \text{Span}\left\{(1,2,3),(0,1,1)\right\},$$

which constitutes a two-dimensional subspace of $\mathbb{R}^3$.

10.2 General Rank-Nullity Theorem

When T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m \times n$ matrix A, the rank-nullity theorem says that $rank(A) + nullity(A) = n$. We want to generalize the idea for arbitrary linear transformations. For general $T : V \to W$, suppose that $\dim[V] = n$ and that $\dim[Ker(T)] = k$. Then k-dimensions worth of the vectors in V are all mapped onto the zero vector in W. Consequently, we only have $n - k$ dimensions worth of vectors left to map onto the remaining vectors in W. This idea gives the following theorem.

Theorem 10.2. *If $T : V \to W$ is a linear transformation and V is finite-dimensional, then*

$$\dim[Ker(T)] + \dim[Ran(T)] = \dim[V].$$

Proof. Omitted. □

Example. Consider the linear transformation $T : P_3(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T(ax^3 + bx^2 + cx + d) = (a + b, b + c, c + d).$$

The kernel of T, denoted as $Ker(T)$, consists of all polynomials $p(x) = ax^3+bx^2+cx+d$ such that $T(p(x)) = (0, 0, 0)$. Setting the output of T equal to the zero vector gives us the system of equations

$$\begin{aligned} a + b &= 0, \\ b + c &= 0, \\ c + d &= 0. \end{aligned}$$

The system gives that $Ker(T) = \{-dx^3 + dx^2 - dx + d | d \in \mathbb{R}\} = span\{-x^3 + x^2 - x + 1\}$.

Since $\dim(Ker(T)) = 1$ and $\dim(P_3(\mathbb{R})) = 4$, by the general rank-nullity theorem, we have $\dim(Ran(T)) = 3$. Since the only subspace of $\mathbb{R}^3$ with 3 dimension is $\mathbb{R}^3$, we get $Ran(T) = \mathbb{R}^3$.

Example. Given a linear transformation $T : V \to W$ with $\dim[V] = n$ and $\text{Ker}(T) = \{0\}$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. We want to show that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for $\text{Ran}(T)$.

We will prove the set is linearly independent. By the general rank-nullity theorem, we know $\dim[Ran(T)] = n$, so we can conclude this linearly independent set is a basis for $Ran(T)$.

Suppose there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) = 0.$$

Since T is linear, we have

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = 0.$$

Because $\text{Ker}(T) = \{0\}$, it implies

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

Since $\{v_1, v_2, \ldots, v_n\}$ are linearly independent, it follows that $c_1 = c_2 = \cdots = c_n = 0$. Hence, $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

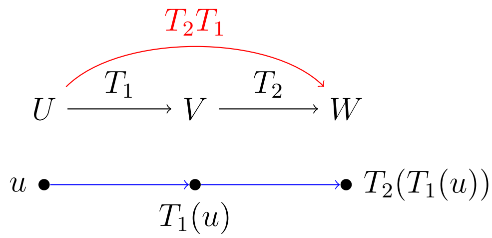
!!! In this example, if $\text{Ker}(T) \neq \{0\}$, there exists a non-zero vector $v \in V$ such that $T(v) = 0$. If v is part of a basis for V, then $T(v)$ would be part of the corresponding set in $\text{Ran}(T)$, but $T(v) = 0$ cannot be part of a basis since it does not contribute to the spanning of $\text{Ran}(T)$ and disrupts linear independence. Thus, if $\text{Ker}(T)$ contains non-zero vectors, the image under T of a basis for V will not necessarily form a basis for $\text{Ran}(T)$.

10.3 Properties of Linear Transformations

The primary goal of this section is to prove that any real vector space of finite dimension n is intrinsically isomorphic to $\mathbb{R}^n$. This necessitates an exploration of the composition of linear transformations.

Definition 10.2. *Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations. We define the* ***composition, or product,*** *$T_2T_1 : U \to W$ by*

$$(T_2T_1)(u) = T_2(T_1(u)) \quad \textit{for all } u \in U.$$



Theorem 10.3. *If $T_1 : U \to V$ and $T_2 : V \to W$ are two linear transformations, then $T_2T_1 : U \to W$ is also a linear transformation.*

Proof. Exercise. □

Examples.

1. Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with matrices A and B, respectively.

   From the definition, for any vector **x** in $\mathbb{R}^n$, we have

   $$(T_2T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x})) = T_2(A\mathbf{x}) = B(A\mathbf{x}) = (BA)\mathbf{x}.$$

   Consequently, T_2T_1 is the linear transformation with matrix BA. Note that A is an $m \times n$ matrix and B is a $p \times m$ matrix, so that the matrix product BA is defined, with size $p \times n$.

2. Let $T_1 : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ and $T_2 : M_n(\mathbb{R}) \to \mathbb{R}$ be the linear transformations defined by

   $$T_1(A) = A + A^T, \quad T_2(A) = \text{tr}(A).$$

   In this case, $T_2T_1 : M_n(\mathbb{R}) \to \mathbb{R}$ is defined by

   $$(T_2T_1)(A) = T_2(T_1(A)) = T_2(A + A^T) = \text{tr}(A + A^T).$$

   This can be written in the equivalent form

   $$(T_2T_1)(A) = 2\text{tr}(A).$$

3. Let $T_1 : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ and $T_2 : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the linear transformations defined by

   $$T_1(A) = A - A^T$$

   and

   $$T_2(A) = A + A^T.$$

   We want to show that T_2T_1 is the zero transformation.

   We compute $T_2(T_1(A))$:

   $$\begin{aligned} T_2(T_1(A)) &= T_2(A - A^T) \\ &= (A - A^T) + (A - A^T)^T \\ &= (A - A^T) + (A^T - (A^T)^T) \\ &= A - A^T + A^T - A \\ &= 0. \end{aligned}$$

The following definitions are very common for any algebraic structures. We focus on linear transformations.

Definition 10.3. *A linear transformation $T : V \to W$ is said to be*

1. ***one-to-one*** *if distinct elements in V are mapped via T to distinct elements in W; that is, whenever $v_1 \neq v_2$ in V, we have $T(v_1) \neq T(v_2)$, or equivalently, whenever $T(v_1) = T(v_2)$, we have $v_1 = v_2$.*

2. ***onto*** *if the range of T is the whole of W; that is, if every $w \in W$ is the image under T of at least one vector $v \in V$.*

3. ***invertible*** *if T is both one-to-one and onto. Then the linear transformation $T^{-1} : W \to V$ defined by*
$$T^{-1}(w) = v \text{ if and only if } w = T(v)$$
*is called the **inverse transformation** to T.*

*Let V and W be vector spaces. If there exists a linear transformation $T : V \to W$ that is invertible, we call T an **isomorphism**, and we say that V and W are **isomorphic vector spaces**, written $V \cong W$.*

Theorem 10.4. *Let $T : V \to W$ be a linear transformation. Then T is one-to-one if and only if Ker$(T) = \{\mathbf{0}\}$.*

Proof. Since T is a linear transformation, we have $T(\mathbf{0}) = \mathbf{0}$. Thus, if T is one-to-one, there can be no other vector v in V satisfying $T(v) = \mathbf{0}$, and so, Ker$(T) = \{\mathbf{0}\}$. Conversely, suppose that Ker$(T) = \{\mathbf{0}\}$. If $v_1 \neq v_2$, then $v_1 - v_2 \neq \mathbf{0}$, and therefore since Ker$(T) = \{\mathbf{0}\}$, $T(v_1 - v_2) \neq \mathbf{0}$. Hence, by the linearity of T, $T(v_1) - T(v_2) \neq \mathbf{0}$, or equivalently, $T(v_1) \neq T(v_2)$. Thus, if Ker$(T) = \{\mathbf{0}\}$, then T is one-to-one. □ □

Remark. In summary, we have the following rules :

$$T : V \to W \text{ is one-to-one } \Leftrightarrow Ker(T) = \{\mathbf{0}\}$$

$$T : V \to W \text{ is onto } \Leftrightarrow Ran(T) = W$$

The following theorem gives the relationship between one-to-one and/or linear transformation from V to W and the dimensions of V and W.

Theorem 10.5. *Let $T : V \to W$ be a linear transformation, and assume that V and W are both finite-dimensional. Then*

1. *If T is one-to-one, then dim$[V] \leq$ dim$[W]$.*

2. *If T is onto, then dim$[V] \geq$ dim$[W]$.*

3. *If T is one-to-one and onto, then dim$[V] =$ dim$[W]$.*

Proof. Exercise. Use the general rank-nullity theorem. □

The last result for this section combines all previous results and provides new characterizations for invertible matrices.

Theorem 10.6. *Let A be an $n \times n$ matrix with real elements, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the matrix transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. The following conditions are equivalent:*

1) A is invertible.

17) T is one-to-one.

18) T is onto.

19) T is an isomorphism.

Proof. By the Invertible Matrix Theorem, A is invertible if and only if nullspace$(A) = \{0\}$. This is equivalent to the statement that Ker$(T) = \{0\}$, and that this is equivalent to the statement that T is one-to-one.

Hence, (1) and (17) are equivalent. Now (17) and (18) are equivalent by the general rank-nullity theorem, and (17) and (18) together are equivalent to (19) by the definition of an isomorphism. □

Examples.

1. $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find Ker$(T)$ and Ran(T), and hence, determine whether the given transformation is one-to-one, onto, both, or neither. If T^{-1} exists, find it.

Solution. We already know $Ran(T) = colspace(A) = span\{(1, 2, 3)\}$. Since this is one dimensional vector space, it cannot be $\mathbb{R}^3$, namely, $Ran(T) \neq \mathbb{R}^3$, so T is not onto. By general rank-nullity theorem $dim(Ker(T)) = 0$ which means $Ker(T) = \{\mathbf{0}\}$, so T is one-to-one. Since T is not onto, it cannot be invertible.

2. $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, find Ker$(T)$ and Ran(T), and hence, determine whether the given transformation is one-to-one, onto, both, or neither. If T^{-1} exists, find it.

Solution. Since $det(A) = 10 \neq 0$, the matrix A is invertible. Therefore, $Ker(T) = nullspace(A) = \{\mathbf{0}\}$ and hence $Ran(T) = colspace(A) = \mathbb{R}^2$. It means that T is both one-to-one and onto, and so T is invertible. Its inverse given by the matrix transformation $A^{-1}\mathbf{x} = \begin{bmatrix} \frac{3}{10} & -\frac{2}{10} \\ -\frac{1}{10} & \frac{4}{10} \end{bmatrix} \mathbf{x}$.

3. $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \end{bmatrix}$, find Ker($T$) and Ran($T$), and hence, determine whether the given transformation is one-to-one, onto, both, or neither. If T^{-1} exists, find it.

Solution. The RREF form of A is $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \end{bmatrix}$. Then $Ran(T) = colspace(A) = \{(1,2),(2,5)\}$, and since this has dimension 2, we get $Ran(T) = \mathbb{R}^2$. Thus, T is onto. On the other hand, $Ker(T) = nullspace(A) = \{(7z, -3z, z) | z \in \mathbb{R}\} \neq \{\mathbf{0}\}$, so T is not one-to-one. Therefore, T is not invertible.

4. Define $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ by

$$T(ax+b) = (2b-a)x + (b+a).$$

Show that T is both one-to-one and onto, and find T^{-1}.

Solution. We have

$$Ker(T) = \{ax+b \mid (2b-a)x + (b+a) = 0\} = \{ax+b \mid (2b-a) = 0 \text{ and } (b+a) = 0\}.$$

It is easy to see that such a and b must be zero, so $Ker(T) = \{\mathbf{0}\}$. By general rank-nullity theorem, $dim(Ran(T)) = 2$. Since $dim(P_1(\mathbb{R})) = 2$, we get $Ran(T) = P_1(\mathbb{R})$. Since T is both one-to-one and onto, it is invertible. The inverse T^{-1} is computed as follows: we know if $T(ax+b) = cx+d$, then $T^{-1}(cx+d) = ax+b$. Form the first equation, we have

$$cx + d = (2b-a)x + (b+a).$$

Therefore, $c = 2b - a$ and $d = b + a$. After solving these, we observe that $b = \frac{c+d}{3}$ and $a = \frac{2d-c}{3}$, i.e.

$$T^{-1}(cx+d) = (\frac{2d-c}{3})x + \frac{c+d}{3}.$$

5. Let V denote the vector space of 2×2 symmetric matrices and define $T : V \to P_2(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = ax^2 + bx + c.$$

Determine whether T is one-to-one, onto, both, or neither. Find T^{-1} or explain why it does not exist.

Solution. It is obvious that $Ker(T) = \{\mathbf{0}\}$, so $dim(Ker(T)) = 0$. Since $dim(V) = 3$, by general rank nullity theorem, we get $dim(Ran(T) = 3$ which gives $Ran(T) = P_2(\mathbb{R})$. Since T is both one-to-one and onto, T is invertible. The inverse T^{-1} is given by:

$$T^{-1}(ax^2 + bx + c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Example. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformations.

(a) Prove that if T_1 and T_2 are both one-to-one, then so is $T_2T_1 : V_1 \to V_3$.

Proof. Assume T_1 and T_2 are both one-to-one. Suppose $T_2T_1(v_1) = T_2T_1(v_2)$. Since T_2 is one-to-one, this implies that $T_1(v_1) = T_1(v_2)$. And since T_1 is also one-to-one, we have $v_1 = v_2$, thus T_2T_1 is one-to-one. □

(b) Prove that if T_1 and T_2 are both onto, then so is $T_2T_1 : V_1 \to V_3$.

Proof. Assume T_1 and T_2 are both onto. Let $w \in V_3$. Since T_2 is onto, there exists a $v_2 \in V_2$ such that $T_2(v_2) = w$. Since T_1 is onto, there exists a $v_1 \in V_1$ such that $T_1(v_1) = v_2$. Therefore, $T_2T_1(v_1) = w$, thus T_2T_1 is onto. □

(c) Prove that if T_1 and T_2 are both isomorphisms, then so is $T_2T_1 : V_1 \to V_3$.

Proof. From parts (a) and (b), if T_1 and T_2 are both one-to-one and onto, then T_2T_1 is both one-to-one and onto, which are the necessary and sufficient conditions for a transformation to be an isomorphism. □

Example. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformations.

(a) Prove that if $T_2T_1 : V_1 \to V_3$ is one-to-one, then so is T_1.

Proof. Exercise. □

(b) Prove that if $T_2T_1 : V_1 \to V_3$ is onto, then so is T_2.

Proof. Exercise. □

We finalize this section with our primary goal:

Theorem 10.7. *If two vector spaces V and W have the same finite dimension, then they are isomorphic.*

Proof. Let V and W be vector spaces over the same field, and suppose that $\dim(V) = \dim(W) = n$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and $\{w_1, w_2, \ldots, w_n\}$ be a basis for W.

Define a linear transformation $T : V \to W$ by mapping the basis vectors of V to the basis vectors of W as follows:

$$T(v_i) = w_i \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Since the basis vectors v_i span V, any vector $v \in V$ can be uniquely expressed as a linear combination of the basis vectors:

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

where $a_1, a_2, \ldots, a_n$ are scalars in the field over which the vector spaces are defined.

The transformation T then maps v to a unique vector in W as follows:

$$T(v) = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = a_1w_1 + a_2w_2 + \cdots + a_nw_n.$$

This map is linear by construction, one-to-one because different vectors in V have different coefficients in the linear combination of basis vectors (and thus map to different vectors in W), and onto because every vector in W can be reached by the image of some vector in V (since the w_i's form a basis for W).

Therefore, T is an isomorphism, and V and W are isomorphic vector spaces. □

Corollary 10.1. *If V is a vector space of dimension n, then V is isomorphic to $\mathbb{R}^n$.*

Proof. Take $W = \mathbb{R}^n$ in the previous theorem. □

10.4 The Matrix of a Linear Transformation

In earlier discussions, we established that any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ corresponds to an $m \times n$ matrix. We now aim to extend this association to arbitrary vector spaces. Let V and W be vector spaces of dimensions n and m, respectively. By selecting ordered bases B for V and C for W, every linear transformation $T : V \to W$ can be uniquely represented by an $m \times n$ matrix. This matrix encapsulates all vital properties of T, encapsulating the essence of linear transformations between finite-dimensional vector spaces within matrix algebra.

Definition 10.4. *Let V and W be vector spaces with ordered bases $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_m\}$, respectively, and let $T : V \to W$ be a linear transformation. The $m \times n$ matrix*

$$[T]_C^B = ([T(v_1)]_C, [T(v_2)]_C, \ldots, [T(v_n)]_C)$$

*is called the **matrix representation of** T relative to the bases B and C. In case $V = W$ and $B = C$, we refer to $[T]_B^B$ simply as the **matrix representation of** T relative to the basis B.*

Example. Let $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ be defined by the transformation

$$T(a + bx + cx^2) = (a - 3c, 2a + b - 2c).$$

We want to find the matrix representation of T with respect to the bases:

(a) $B = \{1, x, x^2\}; C = \{(1, 0), (0, 1)\}$.

(b) $B = \{1, 1 + x, 1 + x + x^2\}; C = \{(1, -1), (2, 1)\}$.

(a) For the basis $B = \{1, x, x^2\}$ and $C = \{(1, 0), (0, 1)\}$, we find the matrix representation of T by applying T to each element of the basis B.

Applying T to 1, x, and x^2 respectively, we get:

$$\begin{aligned} T(1) &= T(1 + 0 \cdot x + 0 \cdot x^2) = (1 - 0, 2 \cdot 1 + 0 - 0) = (1, 2), \\ T(x) &= T(0 + 1 \cdot x + 0 \cdot x^2) = (0 - 0, 2 \cdot 0 + 1 - 0) = (0, 1), \\ T(x^2) &= T(0 + 0 \cdot x + 1 \cdot x^2) = (0 - 3, 2 \cdot 0 + 0 - 2) = (-3, -2). \end{aligned}$$

Thus, the matrix representation of T with respect to bases B and C is:

$$[T]_C^B = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -2 \end{bmatrix}.$$

(b) For the basis $B = \{1, 1 + x, 1 + x + x^2\}$ and $C = \{(1, -1), (2, 1)\}$, we need to express the transformed vectors in terms of the basis C.

We start by applying T to each element of the basis B:

$$\begin{aligned} T(1) &= (1 - 0, 2 + 0 - 0), \\ T(1 + x) &= (1 - 0, 2 + 1 - 0), \\ T(1 + x + x^2) &= (1 - 3, 2 + 1 - 2). \end{aligned}$$

Now we must express each result as a linear combination of vectors in C.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the coordinates of $T(1), T(1+x), T(1+x+x^2)$ in the basis C, respectively. We solve the following system of equations for each transformed vector:

$$\begin{aligned} x_1 \cdot (1, -1) + y_1 \cdot (2, 1) &= (1, 2) & \Rightarrow x_1 &= -1,\ y_1 = 1 \\ x_2 \cdot (1, -1) + y_2 \cdot (2, 1) &= (1, 3) & \Rightarrow x_2 &= -\frac{5}{3},\ y_2 = \frac{4}{3} \\ x_3 \cdot (1, -1) + y_3 \cdot (2, 1) &= (-2, 1) & \Rightarrow x_3 &= -\frac{4}{3},\ y_3 = -\frac{1}{3} \end{aligned}$$

Thus, the matrix representation of T with respect to bases B and C is:

$$[T]_C^B = \begin{bmatrix} 1 & -\frac{5}{3} & -\frac{4}{3} \\ 1 & \frac{4}{3} & -\frac{1}{3} \end{bmatrix}.$$