Linear Algebra & Differential Equations

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12 Week 12

12.1 Ordinary Differential Equations

Definition 12.1. *A differential equation that can be written in the form*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = F(x)$$

*where $a_0, a_1, \ldots, a_n$ are functions of x only, is called a **linear differential equation of order** n.*

Remark. This is linear because our variables are $y, y', y'', \ldots, y^{(n)}$ and they are linear expressions. Note that $y^{(n)}$ is not the same as y^n. The former represents the nth derivative of y, the latter represents the nth power of y. Also we call such equations ordinary because the unknown y depends only on a single variable x.

Examples.

1. $y''' + \cos(x)y'' + e^x y' + 5y = \sinh(x)$ is a linear differential equation of order 3.
2. $y'' + (y')^2 + \sin(x)y = x$ has order 2 but is nonlinear.
3. $xy' + \frac{1}{5+x}y = 0$ is a linear differential equation of order 1.

Exercise: Determine whether the given equations are linear or nonlinear and determine their order.

1. $y = y'$
2. $y''' - y = 5$
3. $y'y'' = 1$
4. $x^5 y^{(4)} - x^3 y'' + 6y = 0$
5. $(1 - x^2)y'' - 4xy' + 5y = \sin x$
6. $xy'' - (y')^2 + y = 0$

Remark. For derivatives we also have the following notations:

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad y''' = \frac{d^3y}{dx^3}, \quad ,\dots, \quad y^{(n)} = \frac{d^ny}{dx^n}$$

Definition 12.2. *A function $y = f(x)$ that is (at least) n times differentiable on an interval I is called a **solution** to the differential equation*

$$a_0(x)y + a_1(x)y' + \cdots + a_n(x)y^{(n)} = f(x) \tag{1}$$

if the substitution $y = f(x), y' = f'(x), \dots, y^{(n)} = f^{(n)}(x)$ satisfies (1) for all x in I.

Examples.

1. The function $y(x) = c_1\cos(2x) + c_2\sin(2x)$ is a solution for

   $$y'' + 4y = 0 \quad \text{on} \quad \mathbb{R}.$$

   First, we calculate the derivatives of $y(x)$:

   $$\begin{aligned} y'(x) &= -2c_1\sin(2x) + 2c_2\cos(2x) \\ y''(x) &= -4c_1\cos(2x) - 4c_2\sin(2x) \end{aligned}$$

   So $y'' + 4y = -4c_1\cos(2x) - 4c_2\sin(2x) + 4(c_1\cos(2x) + c_2\sin(2x)) = 0$.

2. The function $y(x) = c_1x^2\ln x$ is a solution for

   $$x^2y'' - 3xy' + 4y = 0 \quad \text{on} \quad (0,\infty).$$

   Again, we compute the derivatives of $y(x)$:

   $$\begin{aligned} y'(x) &= 2c_1x\ln x + c_1x \\ y''(x) &= 2c_1\ln x + 2c_1 + c_1 = 2c_1\ln x + 3c_1 \end{aligned}$$

   Then, we substitute to verify the equation:

   $$\begin{aligned} x^2y'' - 3xy' + 4y &= x^2\left(2c_1\ln x + 3c_1\right) - 3x(2c_1x\ln x + c_1x) + 4c_1x^2\ln x \\ &= 6c_1x^2\ln x + 3c_1x^2 - 6c_1x^2\ln x - 3c_1x^2 \\ &= 0, \end{aligned}$$

   which confirms that $y(x) = c_1x^2\ln x$ is a solution to the differential equation.

Exercise. Verify that the given function is a solution to a given differential equation and state the maximum interval over which the solution is valid.

1. $y(x) = c_1e^{2x} + c_2e^{-2x}$, $\qquad y'' - 4y = 0$
2. $y(x) = \frac{1}{x+4}$, $\qquad y' = -y^2$

3. $y(x) = e^{ax}(c_1 + c_2 x)$, $\qquad y'' - 2ay' + a^2 y = 0$ where a is constant.

Remark. Nonlinear differential equations also have their solutions written in implicit form, $F(x, y) = 0$, where F defines the solution $y(x)$ implicitly as a function of x. E.g., The relation $x^2 + y^2 - 4 = 0$ defines an implicit solution to the nonlinear differential equation $y' = -\frac{x}{y}$. During the lecture, *our focus is the linear differential equations.*

Definition 12.3. *A solution to an n-th order differential equation on an interval I is called the* ***general solution*** *if it satisfies the following conditions:*

1. *The solution contains a constant $c_1, c_2, \ldots, c_n$.*
2. *All solutions to the equation can be obtained by assigning appropriate values to the constants.*

With fixed constants, the solution is called a ***particular solution****.*

Example. The general solutions to

$$y'' = 18\cos(3x)$$

are of the form

$$y(x) = -2\cos(3x) + c_1 x + c_2.$$

However,

$$y(x) = -2\cos(3x) + x + 5$$

is a particular solution.

Definition 12.4. *An n-th order differential equation together with auxiliary conditions of the form*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants, is called an ***initial-value problem****.*

An example problem is like: Solve the initial-value problem

$$y'' = 18\cos(3x), \quad y(0) = 1, \quad y'(0) = 4.$$

We know $y(x) = -2\cos(3x) + c_1 x + c_2$ is a general solution. Using $y(0) = 1$, we get

$$1 = 2 \cdot 1 + c_1 \cdot 0 + c_2,$$

so $c_2 = 3$. Using $y'(0) = 4$ and $y' = 6\sin(3x) + c_1$, we get

$$4 = 0 + c_1,$$

thus, the solution to the initial-value problem is

$$y(x) = -2\cos(3x) + 4x + 3.$$

Generally we ask:

Solve IVP

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \dots, y_{n-1}$ are constants.

Remark. If $f, y, y', \dots, y^{(n-1)}$ are "good" enough, then the IVP always has a unique solution. See *Picard's Theorem.* The proof is very involved and not our focus. Instead, we will build tools to solve IVPs.

Example. Solve IVP

$$y'' = \cos(x), \quad y(0) = 2, \quad y'(0) = 1.$$

Integrate both sides of $y'' = \cos(x)$ and we get $y' = \sin(x) + c_1$. Integrate again and get $y = -\cos(x) + c_1 x + c_2$.

We have the general solution

$$y(x) = -\cos(x) + c_1 x + c_2.$$

Also use the given initial values, and put

1. $y'(0) = -\sin(0) + c_1 = 1$,
2. $y(0) = -\cos(0) + c_1 0 + c_2 = -1 + c_2 = 2$,

So we get $c_1 = 1, c_2 = 3$. The solution is

$$y(x) = -\cos(x) + x + 3.$$

12.2 First-order Linear Differential Equations

Definition 12.5. *A differential equation that can be written in the form*

$$a(x)\frac{dy}{dx} + b(x)y = c(x),$$

where $a(x), b(x), c(x)$ are functions defined on an interval I is called a first-order linear differential equation. Assuming $a(x) \neq 0$ on I, we can divide both sides and get

$$\frac{dy}{dx} + p(x)y = q(x).$$

Before giving the idea to solve such equations, let's do an example.

Example.

$$\frac{dy}{dx} + y = e^x \tag{2}$$

We want to play with the left part $\left(\frac{dy}{dx} + y\right)$ to imitate product rule.

In other words, we want to have something like

$$f\frac{dy}{dx} + \frac{df}{dx}y$$

so that we can write it as

$$\frac{d}{dx}(fy).$$

To do so let's multiply the equation (2) with e^x and get

$$e^x\frac{dy}{dx} + e^x y = e^{2x} \tag{3}$$

Since $\frac{d}{dx}(e^x) = e^x$, we can write the equation (3) as

$$\frac{d}{dx}(e^x y) = e^{2x}.$$

Now, integrating both sides, we achieve

$$e^x y = \frac{e^{2x}}{2} + c.$$

Thus, we get the following general solution

$$y = \frac{e^x}{2} + \frac{c}{e^x}.$$

The main task is to find the right function to multiply the given equation. This function is called the **integrating factor**.

Now, we'll describe the integrating factor method for general First-order Linear ODEs. We have

$$\frac{dy}{dx} + p(x)y = q(x). \tag{4}$$

Suppose we have a function $I(x)$ such that after multiplying (4) with $I(x)$, we get

$$I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x), \tag{5}$$

and it can be expressed

$$\frac{d}{dx}(I(x)y) = I(x)q(x). \tag{6}$$

Namely, with $I(x)$, we can use the product rule and simplify the given equation (4). Now, we want to compute $I(x)$. To find I we have to make sure that (5) and (6) are the same equations. Since the product rule gives

$$\frac{d}{dx}(I(x)y) = I(x)\frac{dy}{dx} + y\frac{dI}{dx}$$

it means that we have to achieve

$$\frac{dI}{dx} = I(x)p(x). \tag{7}$$

We can solve it as follows, first, divide both sides in (7) with I (assuming $I > 0$) and write the same equation as

$$\frac{1}{I(x)}\frac{dI}{dx} = p(x).$$

Take its integral[1] and get

$$\ln I = \int p(x)dx + c$$

Therefore

$$I = e^{\int p(x)dx + c} = c_1 e^{\int p(x)dx}$$

We need only one $I(x)$ and common scalars can be omitted in (5), it means that we can take $c_1 = 1$, so

$$I(x) = e^{\int p(x)dx}.$$

This is the integrating factor.

General way to solve $y' + p(x)y = q(x)$

1. First, find the integrating factor $I(x) = e^{\int p(x)\,dx}$.
2. Multiply the equation with $I(x)$:
$$I(x)y' + I(x)p(x)y = I(x)q(x),$$
3. Since $I'(x) = I(x)p(x)$, by the product rule, we get
$$[I(x)y]' = I(x)q(x).$$
4. Take the integral of both sides and obtain
$$I(x)y = \int I(x)q(x)\,dx + C.$$
5. Then the solution is
$$y = \frac{1}{I(x)}\left(\int I(x)q(x)\,dx + C\right).$$

[1] Recall that $\int \frac{dI}{I} = ln|I|$ and since we assume $I > 0$, we have $\int \frac{dI}{I} = \ln I$

Examples.

1.
$$x^2y' - 4xy = x^7 \sin x, \quad x > 0.$$

First, write it as
$$y' - \frac{4}{x}y = x^5 \sin x. \tag{8}$$

Then the integrating factor is given by[2]
$$I(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln x + c} = c_1 e^{-4\ln x} = \frac{c_1}{x^4}.$$

So, multiply the equation (8) with $\frac{c_1}{x^4}$ and get
$$\frac{c_1}{x^4}y' - \frac{4c_1}{x^5}y = c_1 x \sin x.$$

Since c_1 can be omitted from both sides, and by the product rule, we get
$$\frac{d}{dx}\left(\frac{1}{x^4}y\right) = x \sin x.$$

Integrate again and get
$$\frac{1}{x^4}y = \int x \sin x \, dx = -x\cos x + \sin x + C$$
$$\Rightarrow y = -x^5 \cos x + x^4 \sin x + Cx^4.$$

2.
$$xy' + 2y = x^2 - x + 1, \quad x > 0.$$

First, write it as
$$y' + \frac{2}{x}y = x - 1 + \frac{1}{x}. \tag{9}$$

Then the integrating factor is given by
$$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x + c} = c_1 e^{2\ln x} = c_1 x^2.$$

So, multiply the equation (9) with c_1x^2 and get
$$c_1x^2y' + 2c_1xy = c_1x^3 - c_1x^2 + c_1x.$$

Since c_1 can be omitted from both sides, and by the product rule, we get
$$\frac{d}{dx}\left(x^2y\right) = x^3 - x^2 + x.$$

[2]We used some properties of $\ln x$ and e^x. For example, $a\ln x = \ln x^a$, and $e^{\ln x} = x$ since $\ln x$ and e^x are inverses of each other

Integrate again and get

$$x^2 y = \int x^3 - x^2 + x \, dx = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\Rightarrow y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2}.$$

3.

$$y' + \frac{2x}{1+x^2} y = \frac{4}{(1+x^2)^2}. \tag{10}$$

Then the integrating factor is given by

$$I(x) = e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)+c} = c_1 e^{\ln(1+x^2)} = c_1(1+x^2).$$

So, multiply the equation (10) with $c_1(1+x^2)$ and get

$$c_1(1+x^2)y' + 2c_1 xy = c_1 \frac{4}{(1+x^2)}.$$

Since c_1 can be omitted from both sides, and by the product rule, we get

$$\frac{d}{dx}\left((1+x^2)y\right) = \frac{4}{(1+x^2)}.$$

Integrate again and get

$$(1+x^2)y = \int \frac{4}{(1+x^2)} \, dx = 4\arctan x + C$$

$$\Rightarrow y = \frac{4\arctan x}{1+x^2} + \frac{C}{1+x^2}.$$

4.

$$y' + \frac{1}{x \ln x} y = 9x^2. \tag{11}$$

Then the integrating factor is given by[3]

$$I(x) = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)+c} = c_1 e^{\ln(\ln x)} = c_1 \ln x.$$

So, multiply the equation (11) with $c_1 \ln x$ and get

$$c_1(\ln x)y' + \frac{c_1}{x} y = c_1(\ln x)9x^2.$$

Since c_1 can be omitted from both sides, and by the product rule, we get

$$\frac{d}{dx}\left((\ln x)y\right) = (\ln x)9x^2.$$

[3] The integral $\int \frac{1}{x \ln x} dx$ can be computed by the substitution $u = \ln x$.

Integrate again and get[4]

$$(\ln x)y = \int (\ln x) 9x^2\, dx = 3x^3(\ln x) - x^3 + C$$

$$\Rightarrow y = 3x^3 - \frac{x^3}{\ln x} + \frac{C}{\ln x}.$$

[4]The right integral follows from the integration by parts.