Linear Algebra & Differential Equations

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13 Week 13

13.1 General Theory for Linear ODEs

We begin with recalling a vector space and a linear transformation.

Let $C^k(I)$ be the set of functions with k continuous derivatives.

This is indeed a subspace of Fun$(I, \mathbb{R})$ because if $f, g \in C^k(I)$ and $c \in \mathbb{R}$,

$$\begin{aligned} (f+g)^{(k)} &= f^{(k)} + g^{(k)} \quad \text{(sum rule for derivative)} \\ (c \cdot f)^{(k)} &= c \cdot f^{(k)}, \end{aligned}$$

We have a particular linear transformation

$$D : C^1(I) \to C^0(I)$$

$$D(f) = f'$$

This is indeed a linear transformation since

$$(a \cdot f + b \cdot g)' = a \cdot f' + b \cdot g'.$$

Now, recall two facts about linear transformations:

1. Composite of linear transformations is again a linear transformation.

2. Linear combinations of linear transformations is again a linear transformation.

Thus, we can define a transformation $D^k : C^k(I) \to C^0(I)$ by composition $D^k = D(D^{k-1})$. Also, if we have $a_1, \ldots, a_n$ scalars, we can get a new linear transformation

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

This transformation actually does

$$\begin{aligned} L(y) &= D^n(y) + a_1 D^{n-1}(y) + \cdots + a_{n-1} D(y) + a_n y \\ &= y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y \end{aligned}$$

Example Let $L = D^3 + 3D^2 - D + 5x$. Then we get $L(y) = y''' + 3y'' - y' + 5xy$. For example, if $y = \cos x$, then

$$\begin{aligned} L(\cos x) &= -\sin x - 3\cos x + \sin x + 5x\cos x \\ &= 2\sin x + (5x - 3)\cos x. \end{aligned}$$

Now consider the general n-th order linear ODE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

where $a_0(x) \neq 0$. We can divide the ODE by a_0 and assume the ODE is in the standard form:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x).$$

Taking $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$, the ODE can be expressed as $L(y) = F(x)$. During the lectures, we assume $a_1, \ldots, a_n, F$ are continuous functions, namely, all ODEs are regular.

The following are important notes about ODEs:

1. If $F(x) = 0$, we have $L(y) = 0$ and we call it **homogeneous ODE**.
2. If $F(x) \neq 0$, we have $L(y) = 0$ and we call it **nonhomogeneous ODE**.
3. If we denote the set of all solutions to the homogeneous ODE by S, we get

   $$S = \{y \in C^n(I) \mid L(y) = 0\} = \ker(L)$$

   This space will be called **the solution space** of the given ODE.
4. The solution space S has dimension n. (It is not an easy fact and needs proof, and it is in the textbook.) Therefore, any set of n linearly independent solutions $\{y_1, \ldots, y_n\}$

   $$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$$

   is a basis for the solution space. So every solution is of the form

   $$c_1 y_1 + \cdots + c_n y_n$$

   where c_i are scalars. This is called the general solution to the ODE.
5. Recall that Wronskian is a nice tool to achieve linear independence of functions.

   Whenever $W(f_1, \ldots, f_n)(x_0) \neq 0$ for some $x_0 \in I$, we get $\{f_1, \ldots, f_n\}$ is linearly independent. If $W(f_1, \ldots, f_n)(x) = 0$ for all $x \in I$, the tool is inconclusive.

   However, if these functions $f_1, \ldots, f_n$ are solutions to an ODE, the Wronskian method works also for dependency.

Theorem 13.1. *Let $y_1, \dots, y_n$ be solutions to the regular nth order ODE $L(y) = 0$ on an interval I. Let $W(y_1, \dots, y_n)(x)$ denote their Wronskian. If $W(y_1, \dots, y_n)(x_0) = 0$ at some point in I, then $\{y_1, \dots, y_n\}$ is linearly dependent.*

Proof. Omitted. □

Zero or nonzero Wronskian on an interval I completely characterizes whether solutions to $L(y) = 0$ are linearly dependent or linearly independent on I.

6. Using the solutions to the homogeneous ODE $L(y) = 0$, we can achieve the solutions to the nonhomogeneous ODE $L(y) = F(x)$.

   Theorem 13.2. *Let $\{y_1, \dots, y_n\}$ be a linearly independent set of solutions to $L(y) = 0$ on an interval I. Let y_p be any particular solution to $L(y) = F(x)$. Then every solution to $L(y) = F(x)$ on I is of the form*

$$y = c_1 y_1 + \cdots + c_n y_n + y_p$$

   for arbitrary constants $c_1, \dots, c_n$.

   Proof. Omitted. □

Summary: For equations

$$y^{(n)} + a_0(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = F(x),$$

we write it as $L(y) = F(x)$ where

$$L = D^n + a_0 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

We will first solve $L(y) = 0$ and it gives the solutions for homogeneous ODEs. Using particular solutions to $L(y) = F(x)$, we achieve all solutions for nonhomogeneous ODEs.

Examples.

1. Consider

$$y''' + x^2 y' - (\sin x) y + e^x y = x^3. \tag{1}$$

   Then define

$$L = D^3 + x^2 D - \sin x D + e^x.$$

   So (1) can be written as

$$L(y) = x^3,$$

   We can solve $L(y) = 0$ first, and find a particular solution to $L(y) = x^3$, Then these together give the general solution to (1).

2. Consider the ODE
$$y'' - 16y = 0.$$
Determine which of the following sets of vectors is a basis for its solution space.
$$S_1 = \{e^{4x}, e^{-4x}\}, \quad S_2 = \{e^{2x}, e^{4x}, e^{-4x}\}, \quad S_3 = \{e^{4x}, e^{2x}\}$$
$$S_4 = \{e^{4x}, e^{-4x}\}, \quad S_5 = \{e^{4x}, 7e^{4x}\}$$
Answer. Since the ODE is of order 2, the solution space is 2-dimensional. So S_1, S_4, S_5 can be a basis.

$$\mathbf{S_3 = \{e^{4x}, e^{2x}\}}$$

First, we need to check if e^{4x}, e^{2x} are solutions.
$$(e^{4x})'' - 16(e^{4x}) = 16e^{4x} - 16e^{4x} = 0 \quad \checkmark$$
$$(e^{2x})'' - 16(e^{2x}) = 4e^{2x} - 16e^{2x} \neq 0 \quad \times$$
So S_3 cannot be a basis.

$$\mathbf{S_4 = \{e^{4x}, e^{-4x}\}}$$

e^{4x} is already a solution,
$$(e^{-4x})'' - 16(e^{-4x}) = (-4)(-4)e^{-4x} - 16e^{-4x} = 0 \quad \checkmark$$
Also
$$W(e^{4x}, e^{-4x}) = \det \begin{pmatrix} e^{4x} & e^{-4x} \\ 4e^{4x} & -4e^{-4x} \end{pmatrix} = -8e^{8x} \neq 0 \text{ if } x = 0,$$
So S_4 is independent, and thus a basis for the solution space.

$$\mathbf{S_5 = \{e^{4x}, 7e^{4x}\}}$$

Since $7e^{4x}$ is a scalar multiple of e^{4x}, S_5 is dependent. So S_5 cannot be a basis.

3. Determine two linearly independent solutions to the $y'' - 7y' + 10y = 0$ of the form $y(x) = e^{rx}$ and determine the general solution

   Answer.
$$\begin{aligned}(e^{rx})'' - 7(e^{rx})' + 10(e^{rx}) &= 0 \\ r^2e^{rx} - 7re^{rx} + 10e^{rx} &= 0 \\ e^{rx}(r^2 - 7r + 10) &= 0.\end{aligned}$$

Since $e^{rx} \neq 0$, we must have $r^2 - 7r + 10 = 0$. Since $r^2 - 7r + 10 = (r-5)(r-2)$, we get $r = 5, 2$. By Wronskian, e^{5x} and e^{2x} are independent. So the general solutions to $y'' - 7y' - 10y = 0$ are of the form $c_1 e^{5x} + c_2 e^{2x}$.

4. Determine two linearly independent solutions to the $x^2y'' + 3xy' - 8y = 0$ of the form $y(x) = x^r$ and determine the general solution on $(0, \infty)$.

   Answer.

$$\begin{aligned} x^2(x^r)'' + 3x(x^r)' - 8x^r &= 0 \\ r(r-1)x^r + 3rx^{r-1} - 8x^r &= 0 \\ (r(r-1) + 3r - 8)x^r &= 0 \\ (r^2 + 2r - 8)x^r &= 0 \end{aligned}$$

   So $r^2 + 2r - 8 = 0$ which means $(r+4)(r-2) = 0$. Thus x^{-4} and x^2 are solutions for the ODE. Use the Wronskian

$$W(x^{-4}, x^2)(x) = \det \begin{pmatrix} x^{-4} & x^2 \\ -4x^{-5} & 2x \end{pmatrix} = 2x^{-3} + 4x^{-3} = 6x^{-3}.$$

   If $x \neq 0$, then $W(x^{-4}, x^2)(x) \neq 0$. So x^{-4} and x^2 are independent. The general solution to $x^2y'' + 3xy' - 8y = 0$ is of the form

$$c_1 x^{-4} + c_2 x^2.$$

5. Determine a particular solution to the given differential equation of the form $y_p(x) = A_0 + A_1 x + A_2 x^2$. Also find the general solution to the differential equation

$$y'' + y' - 2y = 4x^2 + 5.$$

   Answer. Suppose $y_p(x)$ gives a solution, so we must have

$$\begin{aligned} (y_p)'' + (y_p)' - 2(y_p) &= 4x^2 + 5 \\ (2A_2) + (A_1 + 2A_2 x) - 2(A_0 + A_1 x + A_2 x^2) &= 4x^2 + 5 \\ -2A_2 x^2 + (2A_2 - 2A_1)x + (2A_1 + 2A_2 - 2A_0) &= 4x^2 + 5 \end{aligned}$$

   Therefore $A_2 = -2, A_1 = -2, A_0 = -\frac{11}{2}$. For general solution, first we solve

$$y'' + y' - 2y = 0.$$

   Note, when all coefficients are constants, the solutions are of the form e^{rx}.

$$\begin{aligned} (e^{rx})'' + (e^{rx})' - 2(e^{rx}) &= 0 \\ r^2 e^{rx} + re^{rx} - 2e^{rx} &= 0 \\ (r^2 + r - 2)e^{rx} &= 0 \end{aligned}$$

$$\Rightarrow r^2 + r - 1 = (r+2)(r-1) = 0, \text{ namely } r = -2, 1.$$

By Wronskian, it is easy to see that e^{-2x} and e^x are independent.

$\Rightarrow$ the general solution to $y'' + y' - 2y = 0$ is of the form $c_1 e^{-2x} + c_2 e^x$.

$\Rightarrow$ the general solution to $y'' + y' - 2y = 4x + 5$ is of the form

$$c_1 e^{-2x} + c_2 e^x - \frac{11}{2} - 2x - 2x^2.$$

13.2 Constant Coefficient Homogeneous Linear ODEs

In the next few sections, we develop methods for solving linear equations of order n that have only constant coefficients. Namely, our focus is the equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x)$$

where $a_1, \ldots, a_n$ are constants (not functions). First, we'll learn how to solve it when $F(x) = 0$. Second, we'll learn how to solve it for arbitrary $F(x)$.

We begin with writing such an homogeneous ODE using linear transformation. This is given by

$$\mathcal{P}(D)y = 0$$

where $\mathcal{P}(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$. This is called **polynomial differential operator**, and we can write this as a real polynomial

$$\mathcal{P}(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

We will see that solving

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

will be the same as solving $\mathcal{P}(r) = 0$.

Since any polynomial can be expressed as a product of linear factors

$$\mathcal{P}(r) = (r - r_1)^{m_1} (r - r_2)^{m_2} \cdots (r - r_k)^{m_k},$$

we first focus on these factors. Namely, if we have the differential equation

$$(D - r_1)^{m_1} (D - r_2)^{m_2} \cdots (D - r_k)^{m_k} y = 0,$$

we will first learn how to solve

$$(D - r_i)^{m_i} y = 0.$$

Lemma 13.3. *Consider the differential operator $(D - a)^m$, where m is a positive integer and a is a real or complex number. For any $u \in C^m(I)$, we get*

$$(D - a)^m (e^{ax} u) = e^{ax} D^m(u).$$

Proof. If $m = 1$,

$$(D - a)(e^{ax}u) = D(e^{ax}u) - a(e^{ax}u) = e^{ax}D(u) + ae^{ax}u - ae^{ax}u = e^{ax}D(u),$$

By induction, we can repeat the process for higher m. □

Theorem 13.4. *The differential equation $(D - a)^m y = 0$ where m is a positive integer and a is a real or complex number, has the following m solutions that are linearly independent on any interval:*

$$e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}.$$

Proof. Using previous lemma

$$(D - a)^m(x^k e^{ax}) = e^{ax}D^m(x^k) = e^{ax} \cdot 0 = 0.$$

Indeed, since $m > k$, after taking m derivatives of x^k we get 0. Therefore $x^k e^{ax}$ is a solution for any $k = 0, \ldots, m - 1$. Also, they are independent since

$$c_1 e^{ax} + c_2 x e^{ax} + \cdots + c_m x^{m-1} e^{ax} = 0$$

implies (after dividing both sides with e^{ax})

$$c_1 + c_2 x + \cdots + c_m x^{m-1} = 0.$$

Since $\{1, x, \ldots, x^{m-1}\}$ is independent, we get $c_1 = c_2 = \cdots = c_m = 0$. □

Using this theorem, we obtain the general solutions to the differential equation

$$(D - r_1)^{m_1}(D - r_2)^{m_2} \cdots (D - r_k)^{m_k} y = 0.$$

1. For $(D - r)^m$ where r is real, we have independent solutions

$$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}.$$

2. For $(D - r)^m$ where $r = a + ib$, we have independent solutions

$$e^{(a+ib)x}, xe^{(a+ib)x}, \ldots, x^m e^{(a+ib)x}.$$

   Using the fact $e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx)$, observe that we have solutions

$$x^k e^{ax}(\cos bx + i \sin bx) = w_1$$

   and

$$x^k e^{ax}(\cos bx - i \sin bx) = w_2.$$

   Then their linear combinations, as below, give other solutions:

$$\frac{1}{2}(w_1 + w_2) = x^k e^{ax} \cos bx$$

and
$$\frac{1}{2i}(w_1 - w_2) = x^k e^{ax} \sin bx.$$
Therefore, we achieve $2m$ solutions,
$$e^{ax} \cos bx, xe^{ax} \cos bx, \ldots, x^{m-1} e^{ax} \cos bx,$$
$$e^{ax} \sin bx, xe^{ax} \sin bx, \ldots, x^{m-1} e^{ax} \sin bx.$$

3. From the previous two parts, we achieve n linearly independent solutions. Therefore, if $y_1, \ldots, y_n$ are those solutions, the general solution to the equation is given by
$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

Examples.

1. Determine the general solution to the equation
$$y'' - y' - 2y = 0.$$
Solution. Its differential operator is
$$D^2 - D - 2 = (D - 2)(D + 1),$$
So the roots are 2 and -1. Therefore, the general solution to the equation is
$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

2. Determine the general solution to the equation
$$(D + 2)^2 y = 0.$$
Solution. The only root is -2. So the general solution is
$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

3. Determine the general solution to the equation
$$y''' - y'' + y' - y = 0.$$
Solution. Its differential operator is
$$D^3 - D^2 + D - 1 = 0.$$
It can be factorized as
$$(D - 1)(D^2 + 1).$$
The roots are 1 and $\pm i$. Therefore, the general solution is
$$y(x) = c_1 e^x + c_2 e^{0x} \cos x + c_3 e^{0x} \sin x = c_1 e^x + c_2 \cos x + c_3 \sin x.$$

4. Determine the general solution to the equation

$$(D^2+3)(D+1)^2y=0.$$

Solution. The roots are $\pm\sqrt{3}i$ and -1. So the general solution is

$$y(x)=c_1e^{-x}+c_2xe^{-x}+c_3\cos(\sqrt{3}x)+c_4\sin(\sqrt{3}x).$$

5. Determine the general solution to the equation

$$(D^2+2D+10)^2y=0.$$

Solution. The roots are $-1\pm 3i$. Therefore, the general solution is

$$y(x)=c_1e^{-x}\cos(3x)+c_2e^{-x}\sin(3x)+c_3xe^{-x}\cos(3x)+c_4xe^{-x}\sin(3x).$$

6. Determine the general solution to the equation

$$y^{(4)}-16y=0.$$

Solution. Its differential operator is D^4-16. It can be factorized as

$$(D-2)(D+2)(D^2+4),$$

and its roots are $2,-2,\pm 2i$. Thus, the general solution is

$$y(x)=c_1e^{2x}+c_2e^{-2x}+c_3\cos(2x)+c_4\sin(2x).$$

7. Solve the IVP:

$$y''-8y'+16y=0, \quad y(0)=2, \quad y'(0)=7.$$

Solution. Its differential operator is

$$D^2-8D+16=(D-4)^2.$$

So the only root is 4. The general solution is

$$y(x)=c_1e^{4x}+c_2xe^{4x}.$$

Then $y'(x)=4c_1e^{4x}+c_2e^{4x}+4c_2xe^{4x}$. Using the initial values and set

$$y(0)=c_1=2, \quad y'(0)=4c_1+c_2=7.$$

Thus, $c_1=2$ and $c_2=-1$. So the particular solution is

$$y_p(x)=2e^{4x}-xe^{4x}.$$

13.3 The Method of Undetermined Coefficients: Annihilators

In the previous section, we focused on solving constant coefficient homogeneous ODE $P(D)y = 0$. Now we will provide the solution for nonhomogeneous case. Recall the solutions to $P(D)y = F(x)$ are of the form

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is the general solution to $P(D)y = 0$ and $y_p(x)$ is a particular solution to $P(D)y = F(x)$. Thus, our next step is considering how we can find $y_p(x)$.

Before giving the whole process, let's do an example.

Example:

$$(D+5)(D+2)y = 14e^{2x}.$$

Step (1) Solve $(D+5)(D+2)y = 0$. From the previous section, we know the general solution is of the form

$$c_1e^{-5x} + c_2e^{-2x},$$

Step (2) Find a particular solution to $(D+5)(D+2)y = 14e^{2x}$.

Suppose we have another operator $A(D)$ such that $A(D)(14e^{2x}) = 0$. Because then we would have

$$A(D)(D+5)(D+2)y = A(D)(14e^{2x}) = 0,$$

so this would be another homogeneous equation.

It is easy to see that $A(D) = D - 2$. Indeed

$$(D-2)(14e^{2x}) = (14e^{2x})' - 2(14e^{2x}) = 28e^{2x} - 28e^{2x} = 0.$$

Now the general solution for $(D-2)(D+5)(D+2)$ is of the form

$$c_1e^{-5x} + c_2e^{-2x} + A_0e^{2x}.$$

The solution must contain a particular solution to $(D+5)(D+2)y = 14e^{2x}$, we already know $(D+5)(D+2)(c_1e^{-5x} + c_2e^{-2x}) = 0$, so we need to verify

$$(D+5)(D+2)(A_0e^{2x}) = 14e^{2x}.$$

$$(D+5)(D+2)(A_0e^{2x}) = (D^2+7D+10)(A_0e^{2x}) = 4A_0e^{2x} + 14A_0e^{2x} + 10A_0e^{2x} = 14e^{2x}$$
$$\Rightarrow 28A_0e^{2x} = 14e^{2x}$$
$$\Rightarrow A_0 = \frac{1}{2}.$$

Therefore, the general solution to the initial nonhomogeneous ODE is

$$c_1e^{-5x} + c_2e^{-2x} + \frac{e^{2x}}{2}.$$

The trick in this solution is to find $A(D)$ because thanks to $A(D)$ we could achieve the particular solution. This function is called **annihilator** of F. And this whole technique is called "**the method of undetermined coefficients**".

Remark: Note that annihilators satisfies the equation $A(D)F(x) = 0$. Therefore, $F(x)$ can be $c \cdot e^{ax}$, $c \cdot e^{ax}\cos(bx)$, $c \cdot e^{ax}\sin(bx)$, or sum of these. So the annihilator method can be used for such cases. And we have the annihilators for all cases:

1. $A(D) = (D-a)^{k+1}$ annihilates each of
$$e^{ax}, xe^{ax}, \ldots, x^k e^{ax},$$
and their linear combinations.

2. $A(D) = (D^2 - 2aD + a^2 + b^2)^{k+1}$ annihilates each of
$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^k e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^k e^{ax}\sin bx,$$
and their linear combinations.

3. The linear combinations of first type and second type functions are annihilated by the product of individual annihilators.

Examples.

1. $F(x) = 5e^{-3x}$ is annihilated by $A(D) = (D+3)$.
Indeed, $(D+3)(5e^{-3x}) = (5e^{-3x})' + 3(5e^{-3x}) = -15e^{-3x} + 15e^{-3x} = 0$.

2. $F(x) = 2e^x - 3x$.
$2e^x$ is annihilated by $(D-1)$. $-3x = -3xe^0$ is annihilated by $(D-0)^2 = D^2$. Thus, the annihilator of $F(x)$ is $D^2(D-1) = D^3 - D^2$. Indeed,
$$(D^3 - D^2)(2e^x - 3x) = (2e^x - 3x)''' - (2e^x - 3x)'' = 2e^x - 2e^x = 0.$$

3. $F(x) = x^3 e^7 x + 5\sin 4x$.
$x^3 e^7 x$ is annihilated by $(D-7)^4$. $5\sin 4x = 5e^{0x}\sin 4x$ is annihilated by (D^2+16). Thus, the annihilator of $F(x)$ is $(D^2+16)(D-7)^4$.

4. $F(x) = 4e^{-2x}\sin x$ is annihilated by
$$(D^2 - 2(-2)D + (-2)^2 + 1^2)^1 = D^2 + 4D + 5.$$
Exercise. Verify $(D^2+4D+5)(F(x)) = 0$.

5. $F(x) = (1-3x)e^{4x} + 2x^2 = e^{4x} - 3xe^{4x} + 2x^2$
$e^{4x} - 3xe^{4x}$ is annihilated by $(D-4)^2$ and $2x^2$ is annihilated by D^3, so $F(x)$ is annihilated by $D^3(D-4)^2$.

6. $F(x) = e^{4x}(x - 2\sin 5x) + 3x - x^2 e^{-2x}\cos x$

- xe^{4x} is annihilated by $(D-4)^2$.
- $-2e^{4x}\sin 5x$ is annihilated by $D^2 - 8D + 41$.
- $3x$ is annihilated by D^2.
- $-x^2 e^{-2x}\cos x$ is annihilated by $(D^2 + 4D + 5)^3$.

So $F(x)$ is annihilated by

$$(D-4)^2(D^2 - 8D + 41)D^2(D^2 + 4D + 5)^3.$$

Example. Find the general solution to

$$D(D+3)y = 5x + xe^x.$$

First, the general solution to $D(D+3)y = 0$ is

$$c_1 + c_2 e^{-3x}.$$

The annihilator of $5x + xe^x$ is $D^2(D-1)^2$. Now, we have the new homogeneous ODE

$$D^2(D-1)^2 D(D+3)y = D^3(D-1)^2(D+3)y = 0.$$

Its general solution is

$$c_1 + c_2 e^{-3x} + A_0 x + A_1 x^2 + A_2 e^x + A_3 x e^x.$$

Therefore, we expect to have

$$D(D+3)(c_1 + c_2 e^{-3x} + A_0 x + A_1 x^2 + A_2 e^x + A_3 x e^x) = 5x + xe^x,$$

namely, (since $D(D+3)(c_1 + c_2 e^{-3x}) = 0$)

$$D(D+3)(A_0 x + A_1 x^2 + A_2 e^x + A_3 x e^x) = 5x + xe^x.$$

We have

$$\begin{aligned}
& D(D+3)(A_0 x + A_1 x^2 + A_2 e^x + A_3 x e^x) \\
= \; & (A_0 x + A_1 x^2 + A_2 e^x + A_3 x e^x)'' + 3(A_0 x + A_1 x^2 + A_2 e^x + A_3 x e^x)' \\
= \; & (2A_1 + (A_2 + 2A_3)e^x + A_3 x e^x) + (3A_0 + 6A_1 x + 3(A_2 + A_3)e^x + 3A_3 x e^x) \\
= \; & (3A_0 + 2A_1) + (6A_1)x + (4A_2 + 5A_3)e^x + 4A_3 x e^x
\end{aligned}$$

Therefore, $A_1 = \frac{5}{6}$ and $A_3 = \frac{1}{4}$, and so $A_0 = -\frac{5}{9}$ and $A_2 = -\frac{5}{16}$. In other words, the particular solution is

$$-\frac{5x}{9} + \frac{5x^2}{6} - \frac{5e^x}{16} + \frac{xe^x}{4}.$$

The general solution to $D(D+3)y = 5x + xe^x$ is then

$$c_1 + c_2 e^{-3x} - \frac{5x}{9} + \frac{5x^2}{6} - \frac{5e^x}{16} + \frac{xe^x}{4}.$$