Linear Algebra & Differential Equations

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14 Week 14 & 15

More ODE Examples

1. Solve IVP

$$xy' - 2y = x^4 \sin x, \quad y\left(\frac{\pi}{2}\right) = 0$$

Answer: First, write it as

$$y' - \frac{2}{x}y = x^3 \sin x.$$

Integrating factor $I(x)$ is

$$I(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}.$$

Multiplying through by x^{-2}, we get

$$x^{-2}(y' - \frac{2}{x}y) = x^{-2}(x^3 \sin x)$$

Hence,

$$(yx^{-2})' = x \sin x$$

Integrating both sides with respect to x, we get

$$yx^{-2} = \int x \sin x\, dx + C$$

Using integration by parts,

$$\int u\, dv = uv - \int v\, du,$$

let $u = x$ and $dv = \sin x\, dx$, then $du = dx$ and $v = -\cos x$, we have

$$\int x \sin x\, dx = -x \cos x + \int \cos x\, dx = -x \cos x + \sin x + C$$

So the general solution is

$$y = -x^3 \cos x + x^2 \sin x + Cx^2$$

Using the initial condition $y\left(\frac{\pi}{2}\right) = 0$:

$$0 = y\left(\frac{\pi}{2}\right) = 0 + \frac{\pi^2}{4} + C\frac{\pi^2}{4},$$

we find that $C = -1$.

Thus the solution of the IVP is

$$y = -x^3 \cos x + x^2 \sin x - x^2$$

2. Find the general solution to

$$y' + \frac{2x}{x^2 - 1} y = \frac{x}{x^2 - 1}.$$

Answer:

$$I(x) = e^{\int \frac{2x}{x^2-1}\, dx} = e^{\ln u} = u = x^2 - 1,$$

where we let $u = x^2 - 1$ and $du = 2x\, dx$.

Multiplying the differential equation by the integrating factor, we get

$$(x^2 - 1)y' + \frac{2x}{x^2 - 1} y = (x^2 - 1)\left(\frac{x}{x^2 - 1}\right)$$

This simplifies to

$$\big(y(x^2 - 1)\big)' = x.$$

Integrating both sides, we find

$$y(x^2 - 1) = \int x\, dx + C = \frac{x^2}{2} + C,$$

Hence, the general solution is

$$y = \frac{x^2}{2(x^2 - 1)} + \frac{C}{x^2 - 1}.$$

3. Find the general solution to

$$y' + \frac{y}{x \ln x} = x \quad \text{for } x > 1.$$

Answer:

$$I(x) = e^{\int \frac{1}{x \ln x}\, dx} = e^{\ln \ln u} = \ln u = \ln x,$$

where we let $u = \ln x$ and thus $du = \frac{1}{x} dx$.
Multiplying the differential equation by $\ln x$, we get

$$\ln x \left(y' + \frac{y}{x \ln x} \right) = x \ln x.$$

Therefore,

$$(y \ln x)' = x \ln x.$$

Integrating both sides, we get

$$y \ln x = \int x \ln x \, dx + C.$$

Using integration by parts, where $u = \ln x$ and $dv = x\,dx$, then $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$, we find

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x} = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

Finally, the general solution is

$$y = \frac{x^2}{2} - \frac{x^2}{4 \ln x} + \frac{C}{\ln x}.$$

4. Determine the general solution to

$$y'' + 2y' + 5y = 3 \sin 2x,$$

namely,

$$(D^2 + 2D + 5)y = 3 \sin 2x.$$

Step 1: Solve the homogeneous equation

$$(D^2 + 2D + 5)y = 0.$$

The roots are

$$\frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

The general solution is

$$C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x.$$

Step 2: The annihilator of $3 \sin 2x$ is $(D^2 + 4)$. The new ODE is

$$(D^2 + 4)(D^2 + 2D + 5)y = 0.$$

So the general solution is

$$C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x + A_0 \cos 2x + A_1 \sin 2x.$$

We have

$$(D^2 + 2D + 5)(A_0 \cos 2x + A_1 \sin 2x) = 3 \sin 2x$$

which gives

$$(A_0 + 4A_1) \cos 2x + (A_1 - 4A_0) \sin 2x = 3 \sin 2x$$

Solving the system

$$A_0 + 4A_1 = 0$$
$$A_1 - 4A_0 = 3,$$

we find

$$A_1 = \frac{3}{17}, \quad A_0 = -\frac{12}{17}.$$

So the solution is

$$C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x - \frac{12}{17} \cos 2x + \frac{3}{17} \sin 2x.$$

14.1 First-Order Systems

Consider the system of linear differential equations:

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ x_2' &= -a_{21}x_1 + \cdots - a_{2n}x_n + b_2 \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n + b_n, \end{aligned}$$

where $x_1, \ldots, x_n; a_{ij}; b_i$ are all functions with variable t on an interval I. This is called a **first-order system** If $b_i's$ are all 0, then the system is homogeneous. Otherwise, it is nonhomogeneous.

Remarks. Highest derivative is the first derivative. Right sides of equations do not involve any derivative.

Example. Find the general solution to the system

$$\begin{aligned} x_1' &= 2x_1 + x_2, \\ x_2' &= 2x_1 + 3x_2. \end{aligned}$$

The system can be written as

$$\begin{aligned} x_1' - 2x_1 - x_2 = 0 &\Rightarrow (D-2)x_1 - x_2 = 0, \\ x_2' - 2x_1 - 3x_2 = 0 &\Rightarrow -2x_1 + (D-3)x_2 = 0. \end{aligned}$$

Reducing the matrix via $A_{12}(D-3)$

$$A = \begin{pmatrix} D-2 & -1 \\ -2 & D-3 \end{pmatrix}$$

we have

$$\begin{pmatrix} D-2 & -1 \\ (D-2)(D-3)-2 & 0 \end{pmatrix}.$$

The last row means

$$(D^2-5D+4)x_1 = (D-4)(D-1)x_1 = 0.$$

So the general solution is

$$x_1 = C_1 e^{4x} + C_2 e^{x}.$$

Since $x_2 = x_1' - 2x_1$, we get

$$x_2 = 4C_1 e^{4x} + C_2 e^{x} - 2C_1 e^{4x} - 2C_2 e^{x},$$

resulting in

$$x_2 = 2C_1 e^{4x} - C_2 e^{2x}.$$

If we also have initial values, like $x_1(0) = 0$ and $x_2(0) = 3$, we get a system of equations:

$$C_1 + C_2 = 0,$$
$$2C_1 - C_2 = 3.$$

Solving this, we find $C_1 = 1$ and $C_2 = -1$, and the particular solution is

$$x_1 = e^{4x} - e^{x},$$
$$x_2 = 2e^{4x} + e^{x}.$$

Remark. If we have a system involving higher derivatives, we can still rewrite it as a first-order system.

1.

$$x'' - ty = \cos t,$$
$$y'' - x' + x = e^{t}.$$

Let $x_1 = x$, $x_2 = y$, $x_3 = y'$. Then we get

$$x_1' = 0x_1 + tx_2 + 0x_3 + \cos t = tx_2 + \cos t,$$
$$x_2' = 0x_1 + 0x_2 + x_3 + 0 = x_3,$$
$$x_3' = tx_1 + \cos t - x_1 + 0x_3 + e^{t} = -x_1 + tx_1 + (e^{t} + \cos t).$$

2.

$$x'' - 3y' + x = \sin t,$$
$$y'' - tx' - e^{t}y = t^2.$$

Let $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$. Then we get

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= 3x_4 - x_1 + \sin t, \\ x_3' &= x_4, \\ x_4' &= tx_2 + e^t x_3 + t^2. \end{aligned}$$

Example. Solve the given nonhomogeneous system

$$\begin{aligned} x_1' &= x_1 + x_2 + 5e^{4t}, \\ x_2' &= 2x_1 + x_2. \end{aligned}$$

This leads to

$$\begin{aligned} (D-1)x_1 - 2x_2 &= 5e^{4t}, \\ -2x_1 + (D-1)x_2 &= 0. \end{aligned}$$

Apply $A_{12}(\frac{D-1}{2})$ and get

$$\left(\frac{(D-1)^2}{2} - 2\right) x_1 = \frac{(D-1)}{2} 5e^{4t}$$

namely,

$$(D+1)(D-3)x_1 = 15e^{4t}.$$

The solution for x_1 is then

$$x_1 = C_1 e^{3t} + C_2 e^t + 3e^{4t},$$

and for x_2,

$$x_2 = x_1' - x_1 - 5e^{4t} = 3C_1 e^{3t} - C_2 e^t + 12e^{4t}.$$

Hence,

$$x_2 = C_1 e^{3t} - C_2 e^t + 2e^{4t}.$$

14.2 Vector Formulation of First-Order System

Consider the system

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_1(t), \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_n(t). \end{aligned}$$

Let $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $\boldsymbol{x}'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$, and define $A(t) = \left[a_{ij}(t)\right]$, $\boldsymbol{b}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$.

The system can be written as

$$\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t).$$

Let $V_n(I)$ be the set of all column n-vector functions defined on an interval I. $V_n(I)$ is indeed a vector space, but not finite dimensional. Now, to determine the independence of vector functions, we'll generalize the Wronskian definition. If $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t) \in V(I)$, then the Wronskian of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is defined as

$$W(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)(t) = \det \begin{bmatrix} \boldsymbol{x}_1(t) & \cdots & \boldsymbol{x}_n(t) \end{bmatrix}.$$

Again, if $W(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)(a) \neq 0$ at some $a \in I$, we have $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is linearly independent.

Example. 1. Let $\boldsymbol{x}_1(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ and $\boldsymbol{x}_2(t) = \begin{pmatrix} \cot(t) \\ \tan(t) \end{pmatrix}$ be vector functions. They are independent on $(0, \frac{\pi}{2})$ because

$$W(\boldsymbol{x}_1, \boldsymbol{x}_2)(t) = \det \begin{bmatrix} \cos(t) & \cot(t) \\ \sin(t) & \tan(t) \end{bmatrix} = \sin(t) - \cos(t),$$

which is not zero if $t = \frac{\pi}{3}$.

2. The vector functions

$$\boldsymbol{x}_1(t) = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \quad \boldsymbol{x}_2(t) = \begin{pmatrix} 1 \\ t^2 \\ t^4 \end{pmatrix}, \quad \boldsymbol{x}_3(t) = \begin{pmatrix} 1 \\ t^3 \\ t^5 \end{pmatrix}$$

are independent on $(-\infty, \infty)$.

The Wronskian of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ is

$$W(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ t & t^2 & t^3 \\ t^2 & t^4 & t^5 \end{bmatrix} = -t^6 + 2t^5 - t^4.$$

At $t = 2$, this gives us

$$\det = -64 + 64 - 16 = -16 \neq 0.$$

3. Consider another vector function

$$\boldsymbol{x}_1(t) = \begin{pmatrix} \sin^2 t \\ \cos^2 t \\ 2 \end{pmatrix}, \quad \boldsymbol{x}_2(t) = \begin{pmatrix} 2\cos^2 t \\ 2\sin^2 t \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_3(t) = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}.$$

They are dependent on $(-\infty, \infty)$ because

$$2\boldsymbol{x}_1(t) + \boldsymbol{x}_2(t) = \begin{pmatrix} 2\sin^2 t + 2\cos^2 t \\ 2\cos^2 t + 2\sin^2 t \\ 4 + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} = \boldsymbol{x}_3(t).$$

Remark. We have two obvious linear transformations:

1. $T : V_n(I) \to V_n(I)$
$$T(\mathbf{x}(t)) = A(t)\mathbf{x}(t)$$
where $A(t)$ is an $n \times n$ matrix function.

2. $D : V_n(I) \to V_n(I)$
$$D(\mathbf{x}(t)) = \mathbf{x}'(t)$$

So we can express
$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$
as
$$(D - A)\mathbf{x}(t) = \mathbf{b}(t)$$
where D is the first derivative, A is an $n \times n$ matrix function. When $\mathbf{b}(t) = 0$, the solutions $\mathbf{x}(t)$ will be in Ker$(D - A)$. In any case, we have $\mathbf{b}(t) \in$ Ran$(D - A)$.

Our new initial value problem will be like
$$\text{Solve IVP} \begin{cases} \mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t) \\ \mathbf{x}(t_0) = \mathbf{x}_0. \end{cases}$$

When A and $\mathbf{b}$ are continuous functions, we always have a solution (Picard's Theorem again).

Remark. Just as the solution set to a single ODE, the set of solutions to
$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$
is a subspace of $V_n(I)$ and it has dimension n.

Therefore, if $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is any set of n linearly independent solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, then general solutions to it can be written as
$$c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$$
where $c_1, \dots, c_n \in \mathbb{R}$.

Remark. Using the generalized Wronskian, we can have a test for dependency in the case of solutions to $\mathbf{x}' = A\mathbf{x}$.

Theorem 14.1. *If $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a set of solutions to $\mathbf{x}' = A\mathbf{x}$, then ...*

1) If $W(\mathbf{x}_1, \dots, \mathbf{x}_n)(t) \neq 0$ for some $t \in I$, then the set is independent.

2) If $W(\mathbf{x}_1, \dots, \mathbf{x}_n)(t) = 0$ for some $t \in I$, then the set is dependent.

Example. Let
$$\mathbf{x}_1(t) = \begin{bmatrix} t\sin t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} -t\cos t \\ \sin t \end{bmatrix},$$

these are two solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ where $A = \begin{bmatrix} 1/t & t \\ -1/t & 0 \end{bmatrix}$. Indeed,

$$A(t)\mathbf{x}_1(t) = \begin{bmatrix} 1/t & t \\ -1/t & 0 \end{bmatrix} \begin{bmatrix} t\sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} \sin t + t\cos t \\ -\sin t \end{bmatrix} = \mathbf{x}_1'(t)$$

$$A(t)\mathbf{x}_2(t) = \begin{bmatrix} 1/t & t \\ -1/t & 0 \end{bmatrix} \begin{bmatrix} -t\cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} -\cos t + t\sin t \\ \cos t \end{bmatrix} = \mathbf{x}_2'(t)$$

Also, $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent. Indeed,

$$W(\mathbf{x}_1, \mathbf{x}_2)(t) = \det \begin{bmatrix} t\sin t & -t\sin t \\ \cos t & \sin t \end{bmatrix} = t(\sin^2 t + \cos^2 t) = t$$

for $t \neq 0$, $\det \neq 0$.

Remark. Just as we solve non-homogeneous ODEs, when we have non-homogeneous first-order systems, the general solutions to

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

is of the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are independent solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ and $\mathbf{x}_p$ is a particular solution for $\mathbf{b}(t)$.

14.3 First-Order Systems with Nondefective Coefficient Matrices

Recall the simple constant coefficient homogeneous ODE

$$(D - \lambda I)y = 0.$$

We know $y = ce^{\lambda t}$. Instead of single y, consider $\mathbf{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ vector function. Thus

$$(D - \lambda I)\mathbf{x} = 0$$

means

$$\begin{bmatrix} (D - \lambda I)x_1 \\ \vdots \\ (D - \lambda I)x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Using the same approach for each row, we get

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1e^{\lambda t} \\ \vdots \\ a_ne^{\lambda t} \end{bmatrix} = e^{\lambda t} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = e^{\lambda t}\mathbf{v},$$

where $\mathbf{v}$ is the constant vector. It means the vectors are of the form $e^{\lambda t}\mathbf{v}$ where $\mathbf{v}$ is a constant vector and is a solution for $(D - \lambda I)\mathbf{x} = 0$, namely $\mathbf{x}' = \lambda\mathbf{x}$.

A natural question arises: What if λ is an eigenvalue of A and $\mathbf{v}$ is the corresponding eigenvector?

Indeed, if the first-order system

$$\mathbf{x}' = A\mathbf{x}$$

and λ, $\mathbf{v}$ as before, then $e^{\lambda t}\mathbf{v}$ is still a solution:

$$Ae^{\lambda t}\mathbf{v} = e^{\lambda t}(A\mathbf{v}) = e^{\lambda t}\lambda\mathbf{v} = \lambda e^{\lambda t}\mathbf{v} = (e^{\lambda t}\mathbf{v})'.$$

So we have the following result:

Theorem 14.2. *Let A be an $n \times n$ matrix of real constants, and let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$ on any interval.*

Example. Find a solution to

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 1 \end{bmatrix} \mathbf{x}.$$

where the characteristic polynomial of A is

$$p(\lambda) = (\lambda - 1)(\lambda + 2)(\lambda - 3)(\lambda - 4).$$

Solution: $\lambda = 1$ is an eigenvalue of A, and $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 3 & 0 \end{bmatrix} \mathbf{v} = \mathbf{v}$ has a solution $\mathbf{v} = \begin{bmatrix} -y \\ y \\ \frac{2}{3}y \\ \frac{1}{3}y \end{bmatrix}$. For example, $\begin{bmatrix} -3 \\ 3 \\ -2 \\ -1 \end{bmatrix}$ is an eigenvector. So $\mathbf{x} = \begin{bmatrix} -3e^t \\ 3e^t \\ -2e^t \\ -e^t \end{bmatrix}$ is a solution.

Remark: If A has no n independent eigenvectors, namely, A is defective, then there are not enough eigenvectors to get n independent solutions. For nondefective we have all solutions!

Theorem 14.3. *Let A be an $n \times n$ nondefective matrix with independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ corresponding to $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Then the set of vector functions $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ where*

$$\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i, \quad i = 1, \ldots, n$$

are linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$. So any solution is given by $c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$.

Proof. We already know $\mathbf{x}_i$'s are solutions. It's enough to check independency.

$$\begin{aligned} W(\mathbf{x}_1, \ldots, \mathbf{x}_n)(t) &= \det \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & \cdots & e^{\lambda_n t}\mathbf{v}_n \end{bmatrix} \\ &= e^{\lambda_1 t} \det \begin{bmatrix} \mathbf{v}_1 & e^{(\lambda_2-\lambda_1)t}\mathbf{v}_2 & \cdots & e^{(\lambda_n-\lambda_1)t}\mathbf{v}_n \end{bmatrix} \\ &\ldots \\ &= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t} \det \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \neq 0 \end{aligned}$$

since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is independent and $e^{\lambda_1 t} \neq 0$ always. □

Example. Solve $\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & -7 \\ 0 & 2 & -4 \end{bmatrix} \mathbf{x}$, where the characteristic polynomial of A is

$$p(\lambda) = (\lambda - 2)(\lambda + 2)(\lambda - 3).$$

For $\lambda_1 = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -7 \\ 0 & 2 & -6 \end{bmatrix} \mathbf{v} = \mathbf{0} \implies \mathbf{v} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix},$$

For $\lambda_2 = -2$:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & -7 \\ 0 & 2 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0} \implies \mathbf{v} = \begin{bmatrix} 0 \\ y \\ y \end{bmatrix},$$

For $\lambda_3 = 3$:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -7 \\ 0 & 2 & -7 \end{bmatrix} \mathbf{v} = \mathbf{0} \implies \mathbf{v} = \begin{bmatrix} 0 \\ 7z \\ 2z \end{bmatrix}.$$

The solution is of the form

$$c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}.$$

Note that A may have complex eigenvalues and complex eigenvectors. Suppose $\lambda = a + ib$ is a complex eigenvalue and $\mathbf{v} = \mathbf{r} + i\mathbf{s}$ is a complex eigenvector, where $\mathbf{r}$ and s are real vectors. Then, still $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is a solution.

We can expand $e^{\lambda t}\mathbf{v}$ to get real-valued vectors:

$$\begin{aligned} e^{(a+ib)t}\mathbf{v} &= e^{at}(\cos(bt) + i\sin(bt))(\mathbf{r} + i\mathbf{s}) \\ &= e^{at}(\cos(bt)\mathbf{r} - \sin(bt)\mathbf{s}) + ie^{at}(\cos(bt)\mathbf{s} + \sin(bt)\mathbf{r}) \\ &= \mathbf{x}_1 + i\mathbf{x}_2 \end{aligned}$$

We have $\mathbf{x}_1 + i\mathbf{x}_2$ is a solution. Since the conjugate of $\mathbf{v}$ is also an eigenvector (for $a - ib$), we get $\mathbf{x}_1 - i\mathbf{x}_2$ is also a solution. Therefore, by similar computation,

$$\frac{(\mathbf{x}_1 + i\mathbf{x}_2) + (\mathbf{x}_1 - i\mathbf{x}_2)}{2} = \mathbf{x}_1$$

and

$$\frac{(\mathbf{x}_1 + i\mathbf{x}_2) - (\mathbf{x}_1 - i\mathbf{x}_2)}{2i} = \mathbf{x}_2$$

are real solutions. We can summarize the relationship between eigenvectors of A and the solutions to $\mathbf{x}' = A\mathbf{x}$ as follows:

1. If λ is a real eigenvalue of A and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are corresponding independent eigenvectors, we have for $j = 1, \ldots, k$

$$\mathbf{x}_j(t) = e^{\lambda t}\mathbf{v}_j$$

   are all independent solutions to $\mathbf{x}' = A\mathbf{x}$.

2. If $\lambda = a + ib$ is a complex eigenvalue of A and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are corresponding eigenvectors where $\mathbf{v}_j = \mathbf{r}_j + i\mathbf{s}_j$, then

$$\begin{gathered} e^{at}(\cos(bt)\mathbf{r}_1 - \sin(bt)\mathbf{s}_1), \quad e^{at}(\cos(bt)\mathbf{s}_1 + \sin(bt)\mathbf{r}_1), \\ \vdots \\ e^{at}(\cos(bt)\mathbf{r}_k - \sin(bt)\mathbf{s}_k), \quad e^{at}(\cos(bt)\mathbf{s}_k + \sin(bt)\mathbf{r}_k) \end{gathered}$$

   are all independent solutions to $\mathbf{x}' = A\mathbf{x}$,

Example.

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & -1 \\ 0 & 2 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is

$$p(\lambda) = (-\lambda + 3)(\lambda^2 + 4\lambda + 5).$$

Solution: The eigenvalues are

$$\lambda_1 = 3, \quad \lambda_2 = -2 + i, \quad \lambda_3 = -2 - i.$$

For $\lambda_1 = 3$:

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -6 & -1 \\ 0 & 2 & -4 \end{bmatrix} \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_2 = -2 + i$:

$$\begin{bmatrix} 5 - i & 0 & -1 \\ 0 & -1 - i & -1 \\ 0 & 2 & 1 + i \end{bmatrix} \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} \frac{5+i}{26} z \\ \frac{-1+i}{2} z \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{26} z \\ -\frac{1}{2} z \\ z \end{bmatrix} + i \begin{bmatrix} \frac{1}{26} z \\ \frac{1}{2} z \\ 0 \end{bmatrix}.$$

The solutions to $\mathbf{x}' = A\mathbf{x}$ are of the form

$$\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \left(\cos t \begin{bmatrix} 5 \\ -13 \\ 26 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix} \right) + c_3 e^{-2t} \left(\cos t \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 5 \\ -13 \\ 26 \end{bmatrix} \right).$$

Example. Consider the matrix

$$A = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}.$$

We are interested in finding the eigenvalues and eigenvectors of A and the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$.

The characteristic polynomial of A is:

$$\det(\lambda I - A) = \det \begin{bmatrix} -\lambda & -4 \\ 4 & -\lambda \end{bmatrix} = \lambda^2 + 16.$$

Solving for λ, we find the eigenvalues to be $\lambda = \pm 4i$.

For $\lambda = 4i$, the corresponding eigenvector $\mathbf{v}$ is found by solving $(A - 4iI)\mathbf{v} = 0$:

$$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The general solution to the differential equation is then:

$$\mathbf{x}(t) = c_1 \left(\cos(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(4t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 \left(\cos(4t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

Example. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 5 & -1 \\ 1 & 6 & -2 \end{bmatrix}.$$

To find the eigenvalues of A, we calculate the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -1-\lambda & 0 & 0 \\ 1 & 5-\lambda & -1 \\ 1 & 6 & -2-\lambda \end{bmatrix} \\ &= (-1-\lambda)\left[(5-\lambda)(-2-\lambda) + 6\right] \\ &= -(1+\lambda)^2(\lambda - 4). \end{aligned}$$

This gives us the eigenvalues $\lambda_1 = -1$ (with multiplicity 2) and $\lambda_2 = 4$.

For $\lambda_1 = -1$, the corresponding eigenvectors are:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 6 & -1 \\ 1 & 6 & -1 \end{bmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = \begin{bmatrix} -6y + z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ 0 \\ z \end{bmatrix}.$$

For $\lambda_2 = 4$,it gives us

$$\begin{bmatrix} -5 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 6 & -6 \end{bmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = \begin{bmatrix} 0 \\ y \\ y \end{bmatrix}.$$

The solutions to $\mathbf{x}' = A\mathbf{x}$ are of the form

$$\mathbf{x} = c_1 e^{-t} \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$