Linear Algebra & Differential Equations

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4 Week 4 & 5

4.1 Determinant — Intuition

Determinant is a key concept in linear algebra, offering both geometric and algebraic insights. Geometrically, for 2x2 matrices, determinants represent the area of a parallelogram formed by column vectors; in 3x3 matrices, they signify the volume of a parallelepiped. This extends to higher dimensions as a measure of how a linear transformation affects space. Algebraically, determinants play a crucial role in solving systems of linear equations, indicating the existence and uniqueness of solutions.

The recursive nature of determinants refers to the method of expanding a determinant in terms of minors and cofactors (explained in later sections). For a given matrix, its determinant can be calculated by breaking it down into smaller matrices. This is typically done by selecting a row or column, and for each element in that row or column, you multiply the element by the determinant of a smaller matrix, obtained by removing the row and column of that element. The process is repeated recursively for these smaller matrices until reaching 1x1 matrices, where the determinant is simply the element itself.

The recursive nature of determinants is indeed related to their geometric intuition. The process of breaking down a determinant into minors and cofactors can be seen as analyzing the multi-dimensional volume (area in 2D, volume in 3D, etc.) contributed by different parts of the matrix. Each minor represents a submatrix that corresponds to a lower-dimensional parallelotope, and the recursive expansion is akin to decomposing a high-dimensional volume into combinations of lower-dimensional ones.

For a nice discussion about the concept, please read this.

4.2 Determinant — Small Cases

Let A be an $n \times n$ matrix. We will first define what *the determinant of A*, denoted by $\det(A)$, is for small cases $n = 1, 2, 3$. We also provide the geometric idea behind these cases. The ultimate goal is to define the determinant for all n, and we do it *recursively*. It means that we will define the determinant of an $n \times n$ matrix using *smaller matrices living in A*.

The general idea behind geometry of determinant is that $\det(A)$ determines the volume of the object whose coordinates given by the entries of the matrix A. For different dimensions, the volume will have different interpretations.

$\boxed{n=1}$ In this case, we have $A = [a_{11}]$ and its determinant is defined by this unique entry:

$$\det(A) := a_{11}.$$

For example, $\det([3]) = 3$ and $\det([-45]) = -45$. Geometrically, we calculate the volume in one-dimension, which is just *LENGTH*. It is natural to ask "how can the length be negative?" The answer is that the determinant does not just calculate the length of the object, but also determine the orientation of that object. Thus, we can consider that positive length implies the object is on the positive side of the origin, which is zero, while negative length implies the object is on the negative side of the origin.

$\boxed{n=2}$ In this case, we have $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and its determinant is defined as

$$\det(A) := a_{11}a_{22} - a_{12}a_{21}.$$

For example, we have

$$\det\left(\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix}\right) = 2 - 20 = -18 \qquad \det\left(\begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix}\right) = 18 - 8 = 10.$$

Geometrically, we calculate the volume in two-dimension, which is *AREA*, **the area of 2D-object with sides $(\mathbf{a_{11}}, \mathbf{a_{12}})$ and $(\mathbf{a_{21}}, \mathbf{a_{22}})$**. The negative sign again emphasizes the orientation of that 2D-object. If we flip the sides, the orientation then changes. Flipping sides means flipping the rows of the matrix. For example, we have

$$\det\left(\begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}\right) = 20 - 2 = 18 \qquad \det\left(\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}\right) = 8 - 18 = -10.$$

However, when we care only the magnitude of the area, not its orientation, we take the absolute value of the determinant. In other words, we have the following definition.

Definition 4.1. *The area of a parallelogram with sides determined by the vectors (a_{11}, a_{12}) and (a_{21}, a_{22}) is*

$$Area = |det(A)|,$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

$\boxed{n=3}$ In this case, we have $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and its determinant is defined as

$$\det(A) := (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}).$$

It might seem long, but there is a shortcut to obtain this expression. First, add first two columns to the right of the matrix, and draw diagonals in both direction. This is depicted in the following picture. The determinant is calculated by the sum of products of red diagonal entries minus the sum of products of blue diagonal entries.

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

Geometrically, we calculate the volume in three-dimension, which is the usual volume. The discussion about the sign is similar; depending on the orientation, the determinant is $+$ or $-$. When we disregard the orientation, we have the following definition:

Definition 4.2. *The volume of a parallelepiped determined by the vectors* (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}) *and* (a_{31}, a_{32}, a_{33}) *is*

$$\mathit{Volume} = |\mathit{det}(A)|,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Example 1. $\det\left(\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -1 \\ 7 & 0 & 2 \end{bmatrix}\right) = (12 - 28 + 0) - (21 + 0 + 0) = -37.$

Example 2. Consider a parallelepiped in a three-dimensional space defined by the vectors $a = (2, 3, 4)$, $b = (-1, 0, 2)$, and $c = (5, -2, 3)$. Its volume is given by

$$\left|\det\left(\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 2 \\ 5 & -2 & 3 \end{bmatrix}\right)\right| = |(0 + 30 + 8) - (0 - 8 - 9)| = |55| = 55.$$

Example 3. Consider a parallelepiped in a three-dimensional space defined by the vectors $a = (1, 2, 3)$, $b = (2, 4, 6)$, and $c = (-3, -6, -9)$. Its volume is given by

$$\left|\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}\right)\right| = |(-36 - 36 - 36) - (-36 - 36 - 36)| = |0| = 0.$$

Note that when the volume of a 3D object is zero, it means that the object lies on a plane.

4.3 Determinant — General Case, Cofactor Expansions

In the context of an $n \times n$ matrix, minors and cofactors are integral to understanding the determinants and inverses of matrices.

Definition 4.3. *The **minor** of an element in a matrix is the determinant of the smaller matrix formed by removing the row and column of that element. For an element a_{ij} in a matrix A, the minor, denoted as M_{ij}, is obtained by deleting the i-th row and j-th column from A and then taking the determinant of the resulting $(n-1) \times (n-1)$ matrix.*

Definition 4.4. *The **cofactor** of an element is the minor of that element, scaled by $(-1)^{i+j}$. It is denoted as C_{ij} and calculated as $C_{ij} = (-1)^{i+j} M_{ij}$.*

Minors and cofactors are crucial in calculating the determinant of a matrix and in finding the adjugate or adjoint of a matrix, which is essential for computing the inverse of the matrix.

Example 1: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

Let's find the minor and cofactor of the element in the first row and second column ($a_{12} = 2$).

- **Minor**: Remove the first row and second column, leaving the matrix:

  $$\begin{pmatrix} 0 & 5 \\ 1 & 6 \end{pmatrix}.$$

  The minor, M_{12}, is the determinant of this matrix:

  $$M_{12} = 0 \times 6 - 5 \times 1 = -5.$$

- **Cofactor**: The cofactor, C_{12}, is given by:

  $$C_{12} = (-1)^{1+2} \times M_{12} = -1 \times -5 = 5.$$

Example 2: Consider a different matrix

$$B = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let's find the minor and cofactor of the element at $a_{23} = 2$ (second row, third column).

- **Minor**: Remove the second row and third column, forming:

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

  The minor, M_{23}, is:

$$M_{23} = 3 \times 1 - 0 \times 0 = 3.$$

- **Cofactor**: The cofactor, C_{23}, is calculated as:

$$C_{23} = (-1)^{2+3} \times M_{23} = -1 \times 3 = -3.$$

Example 3: Finally, let's use the matrix

$$C = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}.$$

We'll find the minor and cofactor of $a_{31} = 8$ (third row, first column).

- **Minor**: Removing the third row and first column, we get:

$$\begin{pmatrix} 9 & 2 \\ 5 & 7 \end{pmatrix}.$$

  The minor, M_{31}, is:

$$M_{31} = 9 \times 7 - 2 \times 5 = 53.$$

- **Cofactor**: The cofactor, C_{31}, is:

$$C_{31} = (-1)^{3+1} \times M_{31} = 1 \times 53 = 53.$$

The **Cofactor Expansion Theorem**, also known as *Laplace's Expansion*, is a fundamental method used to compute the determinant of a square matrix. It is stated as follows:

Let A be an $n \times n$ matrix with elements a_{ij}. The determinant of A, denoted as det(A), can be calculated by expanding along any row or column using cofactors.

- Expansion along the i-th row:

$$\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} M_{ik} = \sum_{k=1}^{n} a_{ik} C_{ik}$$

  where M_{ik} is the minor of the element a_{ik}, and C_{ik} is the cofactor of a_{ik}.

- Expansion along the j-th column:

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} M_{kj} = \sum_{k=1}^{n} a_{kj} C_{kj}$$

  where M_{kj} is again the minor, and C_{kj} is the cofactor of a_{kj}.

This theorem allows for the calculation of the determinant by breaking down a large matrix into smaller components, and is particularly useful for matrices larger than 2×2, where direct calculation of the determinant becomes more complex.

Example 1: A 3x3 Matrix

Consider the matrix:

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Calculate the determinant of A using cofactor expansion along the first row:

1. Choose the first row for expansion. The determinant of A is:

$$\det(A) = 2C_{11} + 3C_{12} + 1C_{13}$$

where C_{11}, C_{12}, and C_{13} are the cofactors of the elements in the first row.

2. Calculate each cofactor:

$$C_{11} = (-1)^{1+1} \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} = \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

$$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -\det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$$

$$C_{13} = (-1)^{1+3} \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} = \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

3. Compute the determinant:

$$\det(A) = 2 \times \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 3 \times \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 1 \times \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

Example 2: A 4x4 Matrix

Consider the matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

Calculate the determinant of B using cofactor expansion along the second column:

1. Choose the second column for expansion. The determinant of B is:

$$\det(B) = 2C_{12} + 6C_{22} + 10C_{32} + 14C_{42}$$

where $C_{12}, C_{22}, C_{32},$ and C_{42} are the cofactors of the elements in the second column.

2. Calculate each cofactor, which involves computing the determinant of 3x3 matrices (as done in Example 1).

3. Compute the determinant by summing up the products of elements and their corresponding cofactors.

Remark. When given matrix A has a row or column with more zeros than nonzeros, it is better to pick that one to compute determinant with cofactor expansion because the computation would be shorter.

4.4 Properties of Determinant

Note that like we manipulate rows of a matrix, we can do the same for columns. To separate them, let us denote the column operations as follows:

1. CP_{ij} flips the column i and the column j.
2. $CM_i(k)$ multiplies the column i with k.
3. $CA_{ij}(k)$ adds k multiple of the column i to the column j.

Just as elementary matrices corresponded to row operations, we have elementary matrices corresponded to column operations. Using the transpose operation, we have the following obvious results:

$$(CP_{ij})^T = P_{ij}, \qquad (CM_i(k))^T = M_i(k), \qquad (CA_{ij}(k))^T = A_{ij}(k).$$

Let A and B be $n \times n$ matrices. The determinant has the following properties:

P1. If B is obtained by permuting two rows (or columns) of A, then

$$\det(B) = -\det(A).$$

P2. If B is obtained by multiplying any row (or column) of A by a scalar k, then

$$\det(B) = k\det(A).$$

P3. If B is obtained by adding a multiple of any row (or column) of A to another row (or column) of A, then

$$\det(B) = \det(A).$$

P4. For any scalar k, we have
$$\det(kA) = k^n \det(A).$$

P5. We have
$$\det(A^T) = \det(A).$$

P6. Let $r_1, r_2, ..., r_n$ denote the row vectors of A. If the ith row vector of A is the sum of two row vectors, say $r_i = b_i + c_i$, then

$$\det(A) = \det(B) + \det(C),$$

where and $B = \begin{bmatrix} r_1 \\ \vdots \\ r_{i-1} \\ b_i \\ r_{i+1} \\ \vdots \\ r_n \end{bmatrix}$ and $C = \begin{bmatrix} r_1 \\ \vdots \\ r_{i-1} \\ c_i \\ r_{i+1} \\ \vdots \\ r_n \end{bmatrix}$.

The corresponding property for columns is also true.

P7. If A has a row (or column) of zeros, then $\det(A) = 0$.

P8. If two rows (or columns) of A are scalar multiples of one another, then $\det(A) = 0$.

P9. We have
$$\det(AB) = \det(A)\det(B).$$

P10. If A is an invertible matrix, then $\det(A) \neq 0$ and
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

These properties can be very helpful in reducing the amount of work required to evaluate a determinant.

Example 1. Suppose A and B are 4×4 matrices such that $\det(A) = -3$ and $\det(B) = 4$. Then we have following computations using the properties above:

1. $\det(2A) = 2^4 \det(A) = 16(-3) = -48$
2. $\det(A^T B) = \det(A^T)\det(B) = \det(A)\det(B) = (-3)4 = -12$.
3. $\det(B^5) = (\det(B))^5 = 4^5 = 1024$.
4. $\det(B^{-1}AB^T) = \det(B^{-1})\det(A)\det(B^T) = \frac{1}{\det(B)}\det(A)\det(B) = \det(A) = -3$.

Example 2. Let $A = \begin{bmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{bmatrix}$. We can calculate the determinant using cofactor expansion, but to reduce the amount of work, first we reduce the matrix. Applying P_{14}, $A_{12}(-2)$, $A_{13}(-3)$, and $A_{14}(-3)$, we get $B = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & -1 & 4 \\ 0 & 8 & -7 & -1 \end{bmatrix}$. Then using the cofactor expansion using first column, we get

$$\det(B) = b_{11}C_{11} = b_{11}(-1)^{1+1}M_{11} = \det\left(\begin{bmatrix} 3 & 1 & 0 \\ 5 & -1 & 4 \\ 8 & -7 & -1 \end{bmatrix}\right) = 124.$$

Using the rules **P1** and **P3**, we get $\det(A) = -\det(B) = -124$.

Using determinants, we have also the following theoretical results:

Theorem 4.1.

1. *An $n \times n$ matrix A is invertible if and only if det$(A) \neq 0$.*

2. *An $n \times n$ linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution if and only if det$(A) \neq 0$.*

3. *An $n \times n$ homogeneous linear system $A\boldsymbol{x} = \boldsymbol{0}$ has an infinite number of solutions if and only if det$(A) = 0$.*

4.5 Adjoint method for computing A^{-1}

Given a square matrix A of order $n \times n$, **the matrix of cofactors** is formed by the cofactors C_{ij} of each element a_{ij} in matrix A. The cofactor C_{ij} is calculated as:

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

where M_{ij} is the determinant of the submatrix formed by removing the i-th row and j-th column from A.

The **adjoint of matrix** A, denoted as adj(A), is the transpose of the matrix of cofactors. That is, we have

$$\text{adj}(A)_{ij} = C_{ji}$$

Example Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

The matrix of cofactors of A is:

$$\text{Cof}(A) = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

The adjoint of A is:

$$\text{adj}(A) = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

The determinant of A is $\det(A) = 22$. Therefore, the inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \frac{1}{22} \cdot \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

4.6 Cramer's Rule for solving system of equations

Let $A\mathbf{x} = \mathbf{b}$ be an $n \times n$ linear system. If $\text{det}(A) \neq 0$, there is a unique solution, and the solution $(x_1, x_2, \ldots, x_n)$ is given by

$$x_k = \frac{\text{det}(B_k)}{\text{det}(A)}, \quad k = 1, 2, \ldots, n,$$

where B_k denotes the matrix obtained when the kth column vector of A is replaced by b.

Example 1. Given the system of equations:

$$\begin{aligned} x + 5y &= 1, \\ -3x + 6y &= -4, \end{aligned}$$

we use Cramer's Rule to solve for x and y. The determinants are calculated as follows:

$$\begin{aligned} \text{det}(A) = \text{det}\left(\begin{bmatrix} 1 & 5 \\ -3 & 6 \end{bmatrix}\right) &= 1 \cdot 6 - (-3) \cdot 5 = 21, \\ \text{det}(B_x) = \text{det}\left(\begin{bmatrix} \mathbf{1} & 5 \\ \mathbf{-4} & 6 \end{bmatrix}\right) &= 1 \cdot 6 - (-4) \cdot 5 = 26, \\ \text{det}(B_y) = \text{det}\left(\begin{bmatrix} 1 & \mathbf{1} \\ -3 & \mathbf{-4} \end{bmatrix}\right) &= 1 \cdot (-4) - 1 \cdot (-3) = -1. \end{aligned}$$

Using these, we find:

$$\begin{aligned} x &= \frac{\text{det}(B_x)}{\text{det}(A)} = \frac{26}{21}, \\ y &= \frac{\text{det}(B_y)}{\text{det}(A)} = \frac{-1}{21}. \end{aligned}$$

Example 2. Given the system of linear equations:

$$\begin{aligned} 3x - 2y + z &= 4, \\ x + y - z &= 2, \\ x + z &= 1. \end{aligned}$$

We form the coefficient matrix A and the constant matrix b as:

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Using Cramer's Rule, the solutions are given by:

$$x_i = \frac{\det(B_i)}{\det(A)}$$

where A_i is the matrix formed by replacing the i-th column of A with B.

The determinants are calculated as:

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\right) = 6, \\ \det(A_1) &= \det\left(\begin{bmatrix} \mathbf{4} & -2 & 1 \\ \mathbf{2} & 1 & -1 \\ \mathbf{1} & 0 & 1 \end{bmatrix}\right) = 9, \\ \det(A_2) &= \det\left(\begin{bmatrix} 3 & \mathbf{4} & 1 \\ 1 & \mathbf{2} & -1 \\ 1 & \mathbf{1} & 1 \end{bmatrix}\right) = 0, \\ \det(A_3) &= \det\left(\begin{bmatrix} 3 & -2 & \mathbf{4} \\ 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \end{bmatrix}\right) = -3. \end{aligned}$$

Therefore, the solutions are:

$$\begin{aligned} x &= \frac{\det(B_1)}{\det(A)} = \frac{9}{6}, \\ y &= \frac{\det(B_2)}{\det(A)} = 0, \\ z &= \frac{\det(B_3)}{\det(A)} = \frac{-3}{6}. \end{aligned}$$