Linear Algebra & Differential Equations

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If you see any mistake, please email me (euskuplu@usc.edu).

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6 Week 6

Glossary of notations about sets
&
Important notes about mathematical writing

- $\mathbf{x} \in \mathbf{A}$ means that x is an element of the set A. Consider a set as a collection of objects, then an element is a member of that collection.

- $\mathbf{x} \notin \mathbf{A}$ means that x is not an element of the set A.

- $\mathbf{B} \subseteq \mathbf{A}$ means that B is a subset of A, namely, every element of B is also an element of A. Thus, if $y \in B$, then we know also $y \in A$ by subset relation.

- $\emptyset$ means the empty set, namely a set with no elements.

- $\mathbf{B} \cup \mathbf{A}$ means the union of B and A, in other words, if $x \in B \cup A$, then $x \in B$ **or** $x \in A$.

- $\mathbf{B} \cap \mathbf{A}$ means the intersection of B and A, in other words, if $x \in B \cap A$, then $x \in B$ **and** $x \in A$.

- For a set A consider the notation $S = \left\{x \in A \,\middle|\, \text{property}\right\}$ This is a subset of A containing elements x with the **property**. In other words, only the elements satisfying the **property** belong to the subset S. For example $S = \{x \in \mathbb{N} \mid 2 \text{ divides } x\}$ is the set of even natural numbers. Indeed $x \in S$ if and only of 2 divides x, i.e, x is even. Another example $S = \{(a, b, c) \in \mathbb{R}^3 \mid a = b - c\}$ is a subset consists of triples $(a, b.c)$ such that $a = b - c$. Only such vectors belong to S. While $(2, 3, 1) \in S$, we have $(0, 0, 1) \notin S$.

- BE CAREFUL about different sets because they might have different notion. Vectors in $\mathbb{R}^n$, matrices, functions, polynomials, all have different notations. For example, $(a, b, c, d) \in \mathbb{R}^4$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$, and $p(x) = a + bx + cx^2 + dx^3$ seem similar, but they are "literally" different. Therefore, when you are provided V, F, S, etc. first determine the notation, then make your reasoning.

6.1 The plane, $\mathbb{R}^2$

The vector space $\mathbb{R}^2$ consists of all ordered pairs of real numbers, representing the Cartesian plane. As a vector space, $\mathbb{R}^2$ has the following properties:

1. **Addition of Vectors:** For any two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, their sum $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ is also in $\mathbb{R}^2$.
2. **Scalar Multiplication:** For any vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$ and any scalar c in $\mathbb{R}$, the scalar product $c\mathbf{u} = (cu_1, cu_2)$ is in $\mathbb{R}^2$.
3. **Commutativity of Addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$.
4. **Associativity of Addition:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for any vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^2$.
5. **Zero Vector:** The vector $\mathbf{0} = (0, 0)$ serves as the additive identity, meaning $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any vector $\mathbf{u}$ in $\mathbb{R}^2$.
6. **Additive Inverse:** For every vector $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}^2$, there exists an additive inverse $-\mathbf{u} = (-u_1, -u_2)$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
7. **Multiplicative Identity:** For any vector $\mathbf{u}$ in $\mathbb{R}^2$, $1\mathbf{u} = \mathbf{u}$, where 1 is the multiplicative identity in $\mathbb{R}$.
8. **Associativity of Scalar Multiplication:** $c(d\mathbf{u}) = (cd)\mathbf{u}$ for any real numbers c, d and any vector $\mathbf{u}$ in $\mathbb{R}^2$.
9. **Distributivity of Scalar Multiplication with Respect to Vector Addition:** $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for any scalar c and any vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$.
10. **Distributivity of Scalar Multiplication with Respect to Field Addition:** $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ for any scalars c, d and any vector $\mathbf{u}$ in $\mathbb{R}^2$.

Example of Vector Operations on the Cartesian Plane

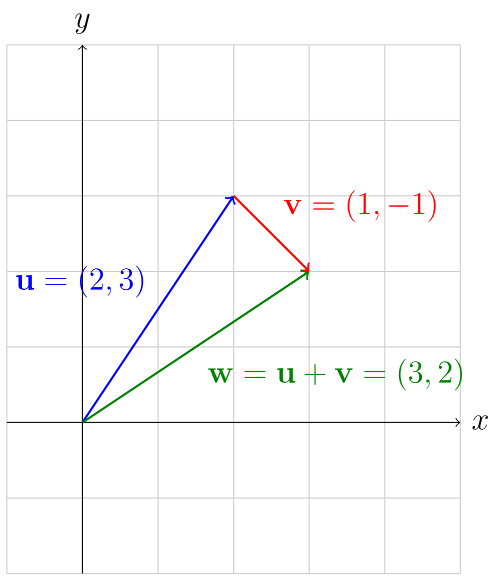
Consider two vectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (1, -1)$ in $\mathbb{R}^2$.

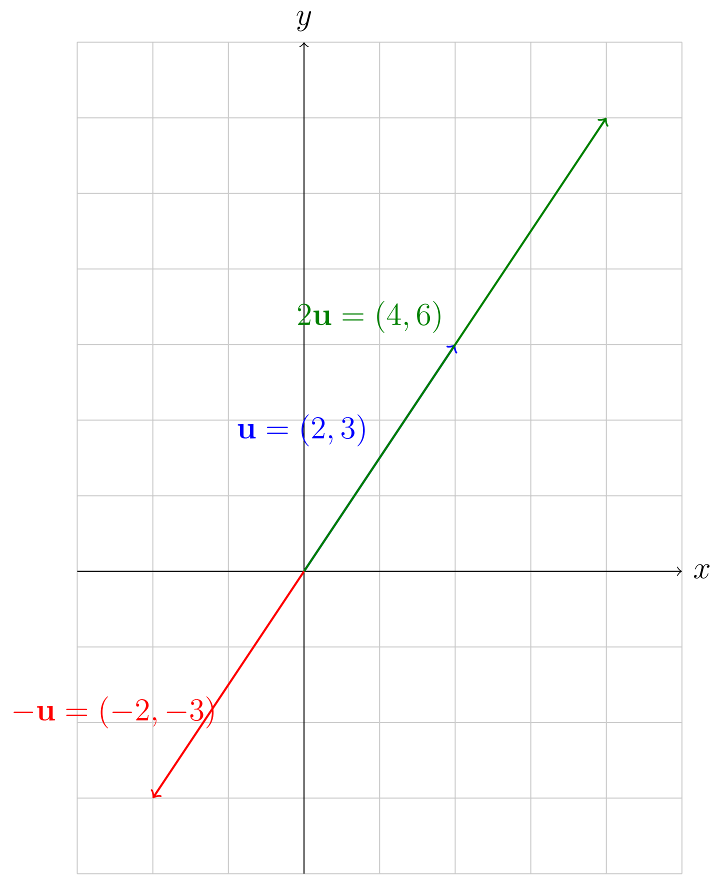
- **Vector Addition:** $\mathbf{u} + \mathbf{v} = (2 + 1, 3 - 1) = (3, 2)$

  This operation visualizes on the Cartesian plane by placing the tail of vector $\mathbf{v}$ at the head of vector $\mathbf{u}$, and drawing a vector from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$. The resulting vector represents the sum $\mathbf{u} + \mathbf{v}$.

- **Scalar Multiplication:** Let $c = 2$, then $c\mathbf{u} = 2(2, 3) = (4, 6)$

  This visualizes by stretching the vector $\mathbf{u}$ to twice its length in the same direction on the Cartesian plane.





6.2 The space, $\mathbb{R}^3$

The vector space $\mathbb{R}^3$ represents all ordered triples of real numbers, visualized in three-dimensional space. Like $\mathbb{R}^2$, $\mathbb{R}^3$ has properties that define its structure as a vector space.

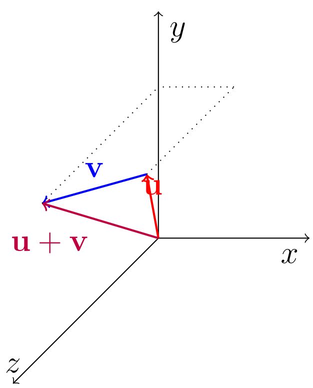
Focusing on the first three properties of the vector space $\mathbb{R}^3$, which are essential to understanding its structure and operations:

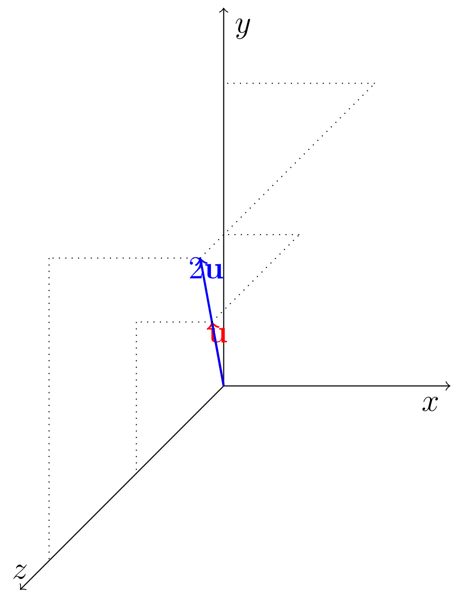
1. **Addition of Vectors**: The process of adding two vectors in $\mathbb{R}^3$ involves combining their corresponding components. If you have two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, their addition is performed as $\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$. This operation reflects the intuitive idea of vector addition as the "tip-to-tail" method in three-dimensional space, where the sum represents the diagonal of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

2. **Scalar Multiplication**: When a vector $\mathbf{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$ is multiplied by a scalar (a real number) c, each component of the vector is multiplied by c, resulting in $c\mathbf{u} = (cu_1, cu_2, cu_3)$. Scalar multiplication can change the magnitude and direction (if $c < 0$) of a vector but keeps it on the same line that passes through the origin and the point represented by the vector.

3. **Zero Vector**: The zero vector in $\mathbb{R}^3$ is defined as $\mathbf{0} = (0, 0, 0)$ and acts as the additive identity in the vector space. This means that adding the zero vector to any vector $\mathbf{u}$ in $\mathbb{R}^3$ leaves $\mathbf{u}$ unchanged, i.e., $\mathbf{u} + \mathbf{0} = \mathbf{u}$. The zero vector represents the origin in three-

dimensional space and has no direction.





6.3 Vector Spaces

Vector spaces are fundamental constructs in linear algebra that generalize the concepts of vector addition and scalar multiplication. The familiar examples of $\mathbb{R}^2$ and $\mathbb{R}^3$ provide intuitive insights into the properties and operations that define a vector space. These examples help in understanding the more abstract concept of vector spaces, including those of higher dimensions, such as $\mathbb{R}^n$.

Properties about operations, evident in $\mathbb{R}^2$ and $\mathbb{R}^3$, are abstracted to define vector spaces in general. The visualization of vectors in these spaces, along with operations such as addition and scalar multiplication, offers a concrete understanding of how vector spaces behave.

The concept of vector spaces extends beyond $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$, where n represents the dimension of the space. In this context, vector spaces can be seen as abstractions of $\mathbb{R}^n$, encompassing any set that satisfies the properties listed above, regardless of the nature of its elements.

In essence, $\mathbb{R}^2$ and $\mathbb{R}^3$ serve as the gateway to understanding vector spaces, demonstrating the operations and properties that define such spaces. They illustrate how vector spaces can be used to model and solve problems in multidimensional settings, providing a foundation upon which the theory of linear algebra is built.

Here is the actual definition of a vector space.

Definition 6.1. *A vector space over a set of scalars F is defined as a set V along with two operations—vector addition and scalar multiplication—satisfying the following ten axioms for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalars a, b in F:*

1. ***Closure under Vector Addition:*** *For any $\mathbf{u}, \mathbf{v} \in V$, the sum $\mathbf{u} + \mathbf{v}$ is also in V.*

2. ***Closure under Scalar Multiplication:*** *For any scalar* $a \in F$ *and any vector* $\mathbf{u} \in V$*, the product* $a\mathbf{u}$ *is in* V*.*

3. ***Commutativity of Vector Addition:***

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

4. ***Associativity of Vector Addition:***

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

5. ***Existence of Additive Identity:*** *There exists an element* $\mathbf{0} \in V$*, called the zero vector, such that*

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

for all $\mathbf{u} \in V$*.*

6. ***Existence of Additive Inverses:*** *For every* $\mathbf{u} \in V$*, there exists a vector* $-\mathbf{u} \in V$ *such that*

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}.$$

7. ***Identity Element of Scalar Multiplication:*** *There exists an element* $1 \in F$*, the multiplicative identity, such that*

$$1\mathbf{u} = \mathbf{u}$$

for all $\mathbf{u} \in V$*.*

8. ***Distributivity of Scalar Multiplication with respect to Vector Addition:***

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

9. ***Distributivity of Scalar Multiplication with respect to Scalar Addition:***

$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}.$$

10. ***Compatibility of Scalar Multiplication with Scalar Multiplication:***

$$a(b\mathbf{u}) = (ab)\mathbf{u}.$$

These axioms establish the structure of a vector space, ensuring that vector addition and scalar multiplication behave in a consistent and predictable manner.

Examples.

1. **Euclidean Spaces:** The set of all n-tuples of real numbers, denoted as $\mathbb{R}^n$, is a vector space under the standard operations of vector addition and scalar multiplication. For instance, $\mathbb{R}^2$ and $\mathbb{R}^3$ represent two-dimensional and three-dimensional spaces, respectively.

2. **Complex Spaces:** The set of all n-tuples of complex numbers, denoted $\mathbb{C}^n$, is a vector space under the usual operations of complex vector addition and scalar multiplication.

3. **Space of Polynomials:** The set of all polynomials of a degree less than or equal to n with coefficients in $\mathbb{R}$, denoted $\mathbb{P}_n$, forms a vector space. Addition and scalar multiplication are performed on the coefficients of the polynomials.

4. **Space of Functions:** The set of all real-valued functions defined on a domain D, denoted $Fun(D, \mathbb{R})$, is a vector space. Here, function addition and scalar multiplication are defined pointwise.

5. **Matrix Spaces:** The set of all $m \times n$ matrices with real entries, denoted $M_{m,n}(\mathbb{R})$, forms a vector space. The operations of matrix addition and scalar multiplication are defined in the usual way.

6. **Space of Continuous Functions:** Denoted by $C[a, b]$, the set of all continuous functions on the interval $[a, b]$ is a vector space under pointwise addition and scalar multiplication.

7. **Space of Differentiable Functions:** The set $C^n[a, b]$ of functions that are n-times differentiable on $[a, b]$ forms a vector space with the usual function addition and scalar multiplication.

8. **Space of Solutions to a Linear Differential Equation:** The set of all solutions to a homogeneous linear differential equation forms a vector space, as the sum of any two solutions and the scalar multiple of any solution are also solutions.

The Space of Polynomials $\mathbb{P}_n$ as a Vector Space

To demonstrate that the space of all polynomials of degree less than or equal to n, denoted by $\mathbb{P}_n$, forms a vector space, we first define polynomial addition and scalar multiplication operations. Then, we verify that $\mathbb{P}_n$ satisfies the ten axioms required for a vector space.

Polynomial Addition: Given two polynomials $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ in $\mathbb{P}_n$, their sum, $(p + q)(x)$, is defined as:

$$(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n.$$

Scalar Multiplication: Given a scalar $c \in \mathbb{R}$ and a polynomial $p(x)$ as above, the scalar product, $(cp)(x)$, is defined as:

$$(cp)(x) = c \cdot a_0 + (c \cdot a_1)x + (c \cdot a_2)x^2 + \ldots + (c \cdot a_n)x^n.$$

To confirm $\mathbb{P}_n$ as a vector space, we must verify it satisfies the following axioms:

1. For any $p(x), q(x) \in \mathbb{P}_n$, the sum $(p + q)(x)$ results in coefficients that are the sum of corresponding coefficients in $p(x)$ and $q(x)$, which are real numbers. Therefore, $(p + q)(x)$ is a polynomial in $\mathbb{P}_n$.

2. Multiplying any $p(x) \in \mathbb{P}_n$ by a scalar $c \in \mathbb{R}$ results in $(cp)(x)$, which is clearly a polynomial of degree less than or equal to n, thus $(cp)(x) \in \mathbb{P}_n$.

3. For $p(x), q(x) \in \mathbb{P}_n$,

$$(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n = (q + p)(x),$$

since real number addition is commutative.

4. For $p(x), q(x), r(x) \in \mathbb{P}_n$,

$$\begin{aligned} &((p + q) + r)(x) \\ &= ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)\, x + \ldots + ((a_n + b_n) + c_n)\, x^n \\ &= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))\, x + \ldots + (a_n + (b_n + c_n))\, x^n \\ &= (p + (q + r))(x), \end{aligned}$$

demonstrating associativity.

5. The zero polynomial $0(x) = 0$ serves as the additive identity because $p(x) + 0(x) = p(x)$ for any $p(x) \in \mathbb{P}_n$.

6. For every $p(x) \in \mathbb{P}_n$, there exists $-p(x) = (-a_0) + (-a_1)x + \ldots + (-a_n)x^n$ such that $p(x) + (-p(x)) = 0(x)$, the zero polynomial.

7. The scalar 1 acts as the multiplicative identity since $1p(x) = p(x)$ for all $p(x) \in \mathbb{P}_n$.

8. For $c \in \mathbb{R}$ and $p(x), q(x) \in \mathbb{P}_n$,

$$c(p + q)(x) = c\left((a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n\right),$$

which, by distributivity of multiplication over addition in $\mathbb{R}$, equals $cp(x) + cq(x)$.

9. For $a, b \in \mathbb{R}$ and $p(x) \in \mathbb{P}_n$,

$$(a + b)p(x) = ((a + b)a_0) + ((a + b)a_1)x + \ldots + ((a + b)a_n)x^n,$$

which equals $ap(x) + bp(x)$ by distributivity in $\mathbb{R}$.

10. For $a, b \in \mathbb{R}$ and $p(x) \in \mathbb{P}_n$,

$$a(bp)(x) = a\left(ba_0 + ba_1x + \ldots + ba_nx^n\right) = (ab)p(x),$$

following the associative property of multiplication in $\mathbb{R}$.

The Space of Functions Fun$(D, \mathbb{R})$ as a Vector Space

To show that the space of all functions from a domain D to $\mathbb{R}$, denoted Fun$(D, \mathbb{R})$, forms a vector space, we define operations of function addition and scalar multiplication, and verify the vector space axioms.

Remark!! In the discussion, we took $D = \mathbb{R}$, but we prove the general case in this note.

Function Addition: Given two functions $f, g \in$ Fun$(D, \mathbb{R})$, their sum, $(f+g)$, is defined by:

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in D.$$

Scalar Multiplication: Given a scalar $c \in \mathbb{R}$ and a function $f \in$ Fun$(D, \mathbb{R})$, the scalar product, (cf), is defined by:

$$(cf)(x) = c \cdot f(x) \quad \text{for all } x \in D.$$

To confirm Fun$(D, \mathbb{R})$ as a vector space, we must verify it satisfies the following axioms:

1. For any $f, g \in$ Fun$(D, \mathbb{R})$, the sum $(f + g)(x)$ is a function in Fun$(D, \mathbb{R})$ because the sum of two real numbers is a real number.

2. Multiplying any $f \in$ Fun$(D, \mathbb{R})$ by a scalar $c \in \mathbb{R}$ results in $(cf)(x)$, which is clearly a function in Fun$(D, \mathbb{R})$.

3. For $f, g \in$ Fun$(D, \mathbb{R})$,

   $$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x),$$

   demonstrating commutativity of addition.

4. For $f, g, h \in$ Fun$(D, \mathbb{R})$,

   $$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ &= f(x) + (g(x) + h(x)) = (f + (g + h))(x), \end{aligned}$$

   showing associativity of addition.

5. The zero function $0(x) = 0$ for all $x \in D$ serves as the additive identity because $f(x) + 0(x) = f(x)$ for any $f \in$ Fun$(D, \mathbb{R})$.

6. For every $f \in$ Fun$(D, \mathbb{R})$, there exists $-f$ defined by $(-f)(x) = -f(x)$ such that $f(x) + (-f)(x) = 0$, the zero function.

7. The scalar 1 acts as the multiplicative identity since $1f(x) = f(x)$ for all $f \in$ Fun$(D, \mathbb{R})$.

8. Scalar multiplication is associative with respect to real number multiplication: For any $a, b \in \mathbb{R}$ and $f \in \text{Fun}(D, \mathbb{R})$, consider $x \in D$. Then,
$$((ab)f)(x) = (ab)f(x) = a(bf)(x) = (a(bf))(x).$$
Thus, $(ab)f = a(bf)$, demonstrating the associativity of scalar multiplication with respect to real number multiplication.

9. For $c \in \mathbb{R}$ and $f, g \in \text{Fun}(D, \mathbb{R})$,
$$c(f + g)(x) = c(f(x) + g(x)) = cf(x) + cg(x),$$
illustrating distributivity of scalar multiplication with respect to function addition.

10. Scalar multiplication distributes over scalar addition: For any $a, b \in \mathbb{R}$ and $f \in \text{Fun}(D, \mathbb{R})$, consider $x \in D$. Then,
$$((a + b)f)(x) = (a + b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x).$$
Thus, $(a + b)f = af + bf$, verifying the distributivity of scalar multiplication over scalar addition.

By demonstrating that $\text{Fun}(D, \mathbb{R})$ satisfies all vector space axioms with function addition and scalar multiplication, we confirm it is indeed a vector space.

6.4 Subspaces

Let V be a vector space over a field F. A nonempty subset S of V is called a subspace of V if and only if it satisfies the following three conditions:

1. Closed under vector addition: For all $u, v \in S$, it holds that $u + v \in S$.
2. Closed under scalar multiplication: For all $u \in S$ and any scalar $a \in F$, it holds that $a \cdot u \in S$.

If these conditions are met, then S is a vector space over F in its own right.

Remark. When $S \subseteq V$ is a subspace, then since S is a vector space, it must have the zero vector. Thus, in practice, if you observe that given S does not contain the zero vector, you can directly conclude that S is not a subspace.

Example. Let V be a vector space over F. Then $S = \{\mathbf{0}\}$, namely, the set containing only the zero vector, is trivially a subspace. It is called **trivial subspace**.

Example. Consider $A \in M_{m \times n}(\mathbb{R})$, namely, A is an $m \times n$ matrix, and consider the subset
$$S = \left\{\mathbf{x} \in \mathbb{R}^n \middle| A\mathbf{x} = \mathbf{0}\right\}.$$
If $\mathbf{x}, \mathbf{y} \in S$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus, $\mathbf{x} + \mathbf{y} \in S$, i.e. S is closed under addition. If $k \in \mathbb{R}$, then $A(k\mathbf{x}) = k(A\mathbf{x}) = k\mathbf{0} = \mathbf{0}$. Thus, $k\mathbf{x} \in S$, i.e S is closed under scalar multiplication. Then S becomes a subspace of $\mathbb{R}^n$, and this is called **null space** of A, and denoted by $\text{nullspace}(A)$.

Examples about $\mathbb{R}^3$

Example 1: $S = \{(a, 0, 0) \big| a \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

Verification for Vector Addition: Let $\mathbf{u} = (u_1, 0, 0)$ and $\mathbf{v} = (v_1, 0, 0)$ be vectors in S. Then, $\mathbf{u} + \mathbf{v} = (u_1 + v_1, 0, 0)$. Since the result is of the form $(a, 0, 0)$ where $a = u_1 + v_1 \in \mathbb{R}$, $\mathbf{u} + \mathbf{v} \in S$.

Verification for Scalar Multiplication: Let $\mathbf{u} = (u_1, 0, 0)$ be a vector in S and let a be any scalar in $\mathbb{R}$. Then, $a \cdot \mathbf{u} = (a \cdot u_1, 0, 0)$. Since the result is of the form $(a \cdot u_1, 0, 0)$ with $a \cdot u_1 \in \mathbb{R}$, $a \cdot \mathbf{u} \in S$.

Therefore, S is closed under vector addition and scalar multiplication, and thus is a subspace of $\mathbb{R}^3$.

Example 2: $S = \{(x, y, z) \big| x + y + z = 0\} \subseteq \mathbb{R}^3$.

Verification for Vector Addition: Consider two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ that lie in S. This means that $u_1 + u_2 + u_3 = 0$ and $v_1 + v_2 + v_3 = 0$.

When we add $\mathbf{u}$ and $\mathbf{v}$, we get $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

To verify that $\mathbf{u} + \mathbf{v}$ also lies in S, we check if their sum satisfies the same equation:

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0.$$

So $\mathbf{u} + \mathbf{v} \in S$, and thus S is closed under vector addition.

Verification for Scalar Multiplication: Let $\mathbf{u} = (u_1, u_2, u_3) \in S$, namely, $u_1 + u_2 + u_3 = 0$, and let a be any scalar.

When we multiply $\mathbf{u}$ by a, we get $a \cdot \mathbf{u} = (a \cdot u_1, a \cdot u_2, a \cdot u_3)$.

To verify that $a \cdot \mathbf{u}$ also lies in S, we check if it satisfies the same equation:

$$a \cdot u_1 + a \cdot u_2 + a \cdot u_3 = a \cdot (u_1 + u_2 + u_3) = a \cdot 0 = 0.$$

So $a \cdot \mathbf{u} \in S$, indicating that S is closed under scalar multiplication.

Example 3: $T = \left\{(a, 2a, 3a) \,\middle|\, a \in \mathbb{R}\right\} \subseteq \mathbb{R}^3$.

Verification for Vector Addition: Let $\mathbf{u} = (u_1, 2u_1, 3u_1)$ and $\mathbf{v} = (v_1, 2v_1, 3v_1)$ be vectors in T. Then, $\mathbf{u} + \mathbf{v} = (u_1 + v_1, 2(u_1 + v_1), 3(u_1 + v_1))$. Since the result is of the form $(a, 2a, 3a)$ where $a = u_1 + v_1 \in \mathbb{R}$, $\mathbf{u} + \mathbf{v} \in T$.

Verification for Scalar Multiplication: Let $\mathbf{u} = (u_1, 2u_1, 3u_1)$ be a vector in T and let a be any scalar in $\mathbb{R}$. Then, $a \cdot \mathbf{u} = (a \cdot u_1, a \cdot 2u_1, a \cdot 3u_1)$. Since the result is of the form $(a \cdot u_1, 2(a \cdot u_1), 3(a \cdot u_1))$ with $a \cdot u_1 \in \mathbb{R}$, $a \cdot \mathbf{u} \in T$.

Therefore, T is closed under vector addition and scalar multiplication, and thus is a subspace of $\mathbb{R}^3$.

Non-example 1: Let $T = \{(a, b, 1) \mid a, b \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

The zero vector in $\mathbb{R}^3$ is $(0, 0, 0)$. However, all vectors in T have their third component fixed at 1, which means T does not contain the zero vector. Thus, T is not a subspace.

Non-example 2: Let $U = \{(a, b, c) \mid a, b, c > 0\} \subseteq \mathbb{R}^3$.

The definition of U requires all components to be positive. Since the zero vector $(0, 0, 0)$ has non-positive (specifically, zero) components, U does not contain the zero vector.

Non-example: Let $I = \{(a, b, c) \mid a, b, c \in \mathbb{Z}\} \subseteq \mathbb{R}^3$, which is the set of all vectors in $\mathbb{R}^3$ whose components are all integers.

Consider a vector $\mathbf{u} = (1, 0, 0)$ in I and a scalar $a = \frac{1}{2} \in \mathbb{R}$ (note that a is not an integer). The product $a \cdot \mathbf{u} = (\frac{1}{2} \cdot 1, \frac{1}{2} \cdot 0, \frac{1}{2} \cdot 0) = (\frac{1}{2}, 0, 0)$ results in a vector that does not have integer components, thus $a \cdot \mathbf{u} \notin I$. Therefore, I cannot be considered a subspace of $\mathbb{R}^3$.

Examples about $M_{3\times3}(\mathbb{R})$

Example 1: The set of all diagonal matrices $D = \{A \in M_{3\times3}(\mathbb{R}) \mid A_{ij} = 0 \text{ for } i \neq j\}$.

Verification for Matrix Addition: Let A and B be matrices in D. Then, $A + B$ is also a diagonal matrix since the sum of two diagonal matrices is diagonal. Therefore, $A+B \in D$.

Verification for Scalar Multiplication: Let A be a matrix in D and let c be any scalar in $\mathbb{R}$. Then, cA is also a diagonal matrix since multiplying a diagonal matrix by a scalar results in another diagonal matrix with each diagonal element scaled by c, and thus $cA \in D$.

Therefore, D is closed under matrix addition and scalar multiplication, and thus is a subspace of $M_{3\times3}(\mathbb{R})$.

Example 2: The set of all symmetric matrices $S = \{A \in M_{3\times3}(\mathbb{R}) \mid A = A^T\}$.

Verification for Matrix Addition: Let A and B be matrices in S. Since $A + B = (A + B)^T = A^T + B^T = A + B$, the sum is also symmetric. Therefore, $A + B \in S$.

Verification for Scalar Multiplication: Let A be a matrix in S and c a scalar in $\mathbb{R}$. Since $cA = (cA)^T = cA^T = cA$, the scalar multiple is also symmetric. Thus, $cA \in S$.

Therefore, S is closed under matrix addition and scalar multiplication, making it a subspace of $M_{3\times3}(\mathbb{R})$.

Example 3: The set of all upper triangular matrices

$$U = \{A \in M_{3\times3}(\mathbb{R}) \mid A_{ij} = 0 \text{ for } i > j\}.$$

Verification for Matrix Addition: Let A and B be matrices in U. Then, $A + B$ is also upper triangular since the sum of two upper triangular matrices is upper triangular. Therefore, $A + B \in U$.

Verification for Scalar Multiplication: Let A be a matrix in U and c a scalar in $\mathbb{R}$. Then, cA is also upper triangular since multiplying an upper triangular matrix by a scalar results in another upper triangular matrix with each element scaled by c, and thus $cA \in U$.

Therefore, U is closed under matrix addition and scalar multiplication, and thus is a subspace of $M_{3\times 3}(\mathbb{R})$.

Non-example 1: The set of all matrices with determinant equal to 1,

$$T = \{A \in M_{3\times 3}(\mathbb{R}) \mid \det(A) = 1\}.$$

Consider two matrices $A, B \in T$ where $A = I_3$ and $B = I_3$. The sum $A + B = 2I_3$ does not have a determinant of 1, thus $A + B \notin T$. Therefore, T is not closed under matrix addition, making it not a subspace of $M_{3\times 3}(\mathbb{R})$.

Non-example 2: The set of all invertible 3×3 matrices,

$$S = \{A \in M_{3\times 3}(\mathbb{R}) \mid \det(A) \neq 0\}.$$

Consider the identity matrix $I_3 \in S$ and a scalar $c = 0$. The product $cI_3 = 0 \cdot I_3$ results in the zero matrix, which is not invertible (i.e., $\det(0I_3) = 0$), thus $cI_3 \notin S$. So S is not a subspace.

Non-example 3: The set of all 3×3 matrices with trace equal to 1,

$$S = \{A \in M_{3\times 3}(\mathbb{R}) \mid \text{tr}(A) = 1\},$$

where tr(A) denotes the trace of matrix A.

This set is not a subspace because: Consider two matrices $A, B \in S$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, both of which have a trace of 1. The sum $A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has a trace of 2, thus $A + B \notin S$.

Examples in $\mathbb{P}_2(\mathbb{R})$

Example 1: The set of all polynomials where the coefficient of x^2 is zero,

$$S = \{a + bx \,|\, a, b \in \mathbb{R}\}.$$

Verification for Closure under Addition: Let $p(x) = a_1 + b_1x$ and $q(x) = a_2 + b_2x$ be in S. Their sum, $p(x) + q(x) = (a_1 + a_2) + (b_1 + b_2)x$, remains a polynomial of degree less than or equal to 1, hence in S.

Verification for Closure under Scalar Multiplication: Let $p(x) = a + bx$ be in S and let c be any scalar in $\mathbb{R}$. Then, $c \cdot p(x) = c(a + bx) = (ca) + (cb)x$, which is still in S.

Therefore, S is a subspace of $\mathbb{P}_2$.

Example 2: The set of all constant polynomials, $T = \{a \,|\, a \in \mathbb{R}\}$.

Verification for Closure under Addition: Let $p(x) = a$ and $q(x) = b$ be in T. Their sum, $p(x) + q(x) = a + b$, is a constant polynomial, hence in T.

Verification for Closure under Scalar Multiplication: Let $p(x) = a$ be in T and let c be any scalar in $\mathbb{R}$. Then, $c \cdot p(x) = c \cdot a = ca$, which is still a constant polynomial, hence in T.

Therefore, T is a subspace of $\mathbb{P}_2$.

Example 3: The set of polynomials that are multiples of x^2, $M = \{cx^2 \mid c \in \mathbb{R}\}$.

Verification for Closure under Addition: Let $p(x) = a_1x^2$ and $q(x) = a_2x^2$ be two polynomials in M. The sum of $p(x)$ and $q(x)$ is given by:

$$p(x) + q(x) = a_1x^2 + a_2x^2 = (a_1 + a_2)x^2.$$

Since $a_1 + a_2$ is a real number, $(a_1 + a_2)x^2$ is also a polynomial in M. Therefore, M is closed under addition.

Verification for Closure under Scalar Multiplication: Let $p(x) = bx^2$ be a polynomial in M and let k be any scalar in $\mathbb{R}$. The scalar multiplication of $p(x)$ by k is given by:

$$k \cdot p(x) = k \cdot (bx^2) = (k \cdot b)x^2.$$

Since $k \cdot b$ is a real number, $(k \cdot b)x^2$ is also a polynomial in M. Therefore, M is closed under scalar multiplication.

Therefore, M is a subspace of $\mathbb{P}_2$.

Non-example 1: The set of all polynomials where the coefficient of x^2 is 1,

$$U = \{a + bx + x^2 \mid b \in \mathbb{R}\}.$$

This set does not include the zero polynomial, which is required for any subspace. Thus, U is not a subspace of $\mathbb{P}_2$.

Non-example 2: The set of polynomials with integer coefficients,

$$V = \{a + bx + cx^2 \mid a, b, c \in \mathbb{Z}\}.$$

Consider $p(x) = 1$ in V and a scalar $c = \frac{1}{2}$. The product $c \cdot p(x) = \frac{1}{2}$ results in a polynomial that does not have integer coefficients, thus $c \cdot p(x) \notin V$. Therefore, V is not a subspace of $\mathbb{P}_2$.

Non-example 3: The set of polynomials with exactly one real root,

$$R = \{a + bx + cx^2 \mid b^2 - 4ac = 0, a, b, c \in \mathbb{R}\}.$$

This set is not a subspace because it is not closed under addition or scalar multiplication. For instance, the polynomials $p(x) = 2 + 4x + 2x^2$ and $q(x) = 1 - 2x + x^2$ both have exactly one real root, but their sum, $p(x) + q(x) = 3 + 2x + 3x^2$, does not necessarily have exactly one real root. Thus, R fails the closure under addition criterion.

Non-example 4: The set of polynomials that are positive for all real numbers x, $P = \{a + bx + cx^2 \,|\, \forall x \in \mathbb{R}, a + bx + cx^2 > 0\}$.

The set P is not a subspace of $\mathbb{P}_2$ because it does not include the zero polynomial, which is a requirement for any subspace. Moreover, P is not closed under scalar multiplication by negative numbers. For example, $p(x) = 1+x^2$ is always positive, but multiplying by -1 yields $-1 - x^2$, which is not positive for all x, thus failing the closure under scalar multiplication criterion.

Examples and Non-examples within Fun$(\mathbb{R}, \mathbb{R})$

Example 1: The set of all constant functions $C = \{f \mid f(x) = c \text{ for some } c \in \mathbb{R}\}$.

Verification for Function Addition: Let $f, g \in C$ such that $f(x) = a$ and $g(x) = b$ for all $x \in \mathbb{R}$. Their sum, $(f + g)(x) = f(x) + g(x) = a + b$, is also a constant function, hence $(f + g) \in C$.

Verification for Scalar Multiplication: Let $f \in C$ with $f(x) = a$ and let $\alpha \in \mathbb{R}$. Then, $(\alpha f)(x) = \alpha f(x) = \alpha a$ is also a constant function, thus $(\alpha f) \in C$.

Example 2: The set of all linear functions $L = \{f \mid f(x) = mx + b, m, b \in \mathbb{R}\}$.

Verification for Function Addition: Let $f, g \in L$ where $f(x) = mx+b$ and $g(x) = nx+c$. Their sum, $(f + g)(x) = (m + n)x + (b + c)$, is also a linear function, so $(f + g) \in L$.

Verification for Scalar Multiplication: Let $f \in L$ with $f(x) = mx + b$ and let $\alpha \in \mathbb{R}$. Then, $(\alpha f)(x) = \alpha(mx+b) = (\alpha m)x+(\alpha b)$, which is also a linear function, hence $(\alpha f) \in L$.

Example 3: The set of all even functions $E = \{f \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$.

Verification for Function Addition: Let $f, g \in E$. Then, for all $x \in \mathbb{R}$, $(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x)$, hence $(f + g) \in E$.

Verification for Scalar Multiplication: Let $f \in E$ and let $\alpha \in \mathbb{R}$. Then, for all $x \in \mathbb{R}$, $(\alpha f)(x) = \alpha f(x) = \alpha f(-x) = (\alpha f)(-x)$, hence $(\alpha f) \in E$.

Non-example 1: The set of exponential functions with positive bases,

$$EXP = \{f \mid f(x) = a^x, a > 1\}$$

Consider two functions $f(x) = 2^x$ and $g(x) = 3^x$ in EXP. Their sum, $(f + g)(x) = 2^x + 3^x$, does not have the form a^x with a single base $a > 1$, hence $(f + g) \notin EXP$. So EXP is not a subspace.

Non-example 2: The set of functions where $f(0) = 1$,

$$F_1 = \{f \mid f(0) = 1\}.$$

While this set contains functions that satisfy a specific condition at $x = 0$, it does not contain the zero function, $f(x) = 0$, since $f(0) \neq 1$ for the zero function. This violates the requirement for containing the zero vector (function) in the subspace criteria.

Non-example 3: The set of functions that intersect the y-axis at a point greater than -1,

$$H = \{f \mid f(0) > -1\}.$$

Let $f \in H$ with $f(0) = 1$ and consider a scalar product $-2f$. Since the function satisfies $(-2f)(0) = -2 < -1$, so H is not closed under scalar multiplication.

6.5 Spanning Sets

Definition 6.2. *A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V over a field F is an expression of the form*

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars in F.

This concept is fundamental in linear algebra and is used to describe the span of a set of vectors, which is the set of all possible linear combinations of those vectors.

Example.

Consider the vectors $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (3, -1)$ in the vector space $\mathbb{R}^2$. A linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ can be expressed as

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a(1, 2) + b(3, -1) = (a + 3b, 2a - b),$$

where $a, b \in \mathbb{R}$. For example, if $a = 2$ and $b = 1$, then the linear combination is

$$2\mathbf{v}_1 + 1\mathbf{v}_2 = (2 + 3 \cdot 1, 2 \cdot 2 - 1) = (5, 3).$$

Example.

Consider the polynomials $p_1(x) = x^2$, $p_2(x) = x$, and $p_3(x) = 1$ in the vector space of polynomials of degree at most 2, denoted as $P_2(\mathbb{R})$. A linear combination of p_1, p_2, and p_3 can be written as

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = c_1x^2 + c_2x + c_3,$$

where $c_1, c_2, c_3 \in \mathbb{R}$. For instance, with $c_1 = 1$, $c_2 = -2$, and $c_3 = 3$, the linear combination becomes

$$1 \cdot x^2 - 2 \cdot x + 3 = x^2 - 2x + 3,$$

which is another polynomial in $P_2(\mathbb{R})$.

Definition 6.3. *The **span** (or the set **spanned by** a set of vectors) of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V over a field F is defined as the set of all possible linear combinations of these vectors. Formally, the span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is given by*

$$\textit{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \mid \alpha_i \in F, \textit{ for all } i = 1, 2, \ldots, n\}.$$

This concept describes the subset of V that can be constructed by linearly combining the given vectors.

Examples.

1. Consider vectors $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$ in $\mathbb{R}^3$. The span of these vectors is

   $$\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \{a\mathbf{v}_1 + b\mathbf{v}_2 \mid a, b \in \mathbb{R}\} = \{(a, b, 0) \mid a, b \in \mathbb{R}\},$$

   which represents all vectors in the xy-plane of $\mathbb{R}^3$.

2. Let $p_1(x) = 1$, $p_2(x) = x$, and $p_3(x) = x^2$ in the vector space of polynomials of degree at most 3, $P_3(\mathbb{R})$. The span of these polynomials is

   $$\text{Span}(\{p_1, p_2, p_3\}) = \{ap_1(x)+bp_2(x)+cp_3(x) \mid a, b, c \in \mathbb{R}\} = \{a+bx+cx^2 \mid a, b, c \in \mathbb{R}\},$$

   which includes all polynomials of degree at most 2.

3. Consider the functions $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$ in the space of continuous functions over $\mathbb{R}$, denoted by $C(\mathbb{R})$. The span of f_1 and f_2 is

   $$\text{Span}(\{f_1, f_2\}) = \{af_1(x) + bf_2(x) \mid a, b \in \mathbb{R}\} = \{a\cos(x) + b\sin(x) \mid a, b \in \mathbb{R}\},$$

   which represents all linear combinations of $\cos(x)$ and $\sin(x)$.

4. Consider the matrices

   $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   in the vector space of 2×2 matrices over $\mathbb{R}$, $M_{2\times2}(\mathbb{R})$. The span of these matrices is

   $$\text{Span}(\{\mathbf{A}, \mathbf{B}\}) = \{a\mathbf{A} + b\mathbf{B} \mid a, b \in \mathbb{R}\} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\},$$

   which represents all 2×2 matrices where the first row can be any real numbers a and b, and the second row is always zero.

Theorem 6.1. *The span of any set of vectors in a vector space V over a field F is a subspace of V.*

Proof. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors in V, and let Span(S) denote the span of S. To prove that Span(S) is a subspace of V, we must show that it satisfies the following three criteria:

1. *Closed Under Vector Addition*: For any vectors $\mathbf{u}, \mathbf{v} \in$ Span(S), there exist scalars $a_i, b_i \in F$ such that $\mathbf{u} = \sum_{i=1}^k a_i\mathbf{v}_i$ and $\mathbf{v} = \sum_{i=1}^k b_i\mathbf{v}_i$. The sum $\mathbf{u} + \mathbf{v} = \sum_{i=1}^k (a_i + b_i)\mathbf{v}_i$ is also a linear combination of vectors in S, implying that $\mathbf{u} + \mathbf{v} \in$ Span(S).

2. *Closed Under Scalar Multiplication*: For a vector $\mathbf{u} \in$ Span(S) and any scalar $c \in F$, if $\mathbf{u} = \sum_{i=1}^k a_i\mathbf{v}_i$, then $c\mathbf{u} = c\sum_{i=1}^k a_i\mathbf{v}_i = \sum_{i=1}^k (ca_i)\mathbf{v}_i$ is also a linear combination of vectors in S, hence $c\mathbf{u} \in$ Span(S).

Therefore, Span(S) satisfies all the criteria for being a subspace of V. □

Remark. This theorem underscores the foundational aspect of span in linear algebra, illustrating how the span of a set of vectors forms the smallest subspace containing all those vectors. It is a crucial concept for understanding vector spaces, as it provides a method for constructing new subspaces and for studying the properties of existing ones.

Definition 6.4. *A **spanning set** for a vector space V is a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in V such that every vector in V can be expressed as a linear combination of these vectors. In other words, if the span of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is equal to V, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for V. Formally,*

$$V = \mathit{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}).$$

A spanning set provides a way to construct any vector in the space from its elements, emphasizing the idea of generating the entire vector space.

Examples.

1. **Spanning Set in $\mathbb{R}^2$:** The set $\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$ forms a spanning set for $\mathbb{R}^2$. Any vector $\mathbf{v} = (a,b)$ in $\mathbb{R}^2$ can be expressed as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2.$$

2. **Spanning Set for Polynomials of Degree at Most 2:** In $P_2(\mathbb{R})$, the set $\{1, x, x^2\}$ is a spanning set. Any polynomial $p(x) = ax^2 + bx + c$, with $a, b, c \in \mathbb{R}$, can be written as:

$$p(x) = c \cdot 1 + b \cdot x + a \cdot x^2.$$

3. **Spanning Set for 2×2 Matrices $M_{2\times 2}(\mathbb{R})$:** Consider the vector space of 2×2 matrices over $\mathbb{R}$. The set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

forms a spanning set for $M_{2\times 2}(\mathbb{R})$. Any 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be expressed as a linear combination of these matrices:

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

More examples related to $\mathbb{R}^n$.

1. Consider the set of vectors $\{\mathbf{v}_1 = (2,1), \mathbf{v}_2 = (1,1)\}$ in $\mathbb{R}^2$. We aim to show that this set spans $\mathbb{R}^2$, meaning any vector $\mathbf{v} = (a,b) \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

To express $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, we write:

$$x\mathbf{v}_1 + y\mathbf{v}_2 = x(2,1) + y(1,1) = (a,b),$$

which leads to the system of equations:

$$\begin{aligned} 2x + y &= a, \\ x + y &= b. \end{aligned}$$

The matrix equation is then

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

To find $\mathbf{x}$, we solve the equation using the inverse of A, provided A is invertible. The inverse of A is

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

since $\det(A) = (2)(1) - (1)(1) = 1$.

Multiplying both sides of the equation by A^{-1} gives us

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ -a+2b \end{pmatrix}.$$

This shows that for any vector (a,b) in $\mathbb{R}^2$, there exist coefficients $x = a - b$ and $y = -a + 2b$ such that $x\mathbf{v}_1 + y\mathbf{v}_2 = (a,b)$. Therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans $\mathbb{R}^2$.

2. Given the vectors $\mathbf{v}_1 = (1,0,1)$, $\mathbf{v}_2 = (0,1,1)$, and $\mathbf{v}_3 = (1,1,0)$ in $\mathbb{R}^3$, we aim to show that any vector $\mathbf{v} = (a,b,c)$ can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. This is accomplished by solving the matrix equation

$$A\mathbf{x} = \mathbf{b},$$

where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

The matrix equation becomes

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

To solve for $\mathbf{x}$, we first determine if A is invertible by calculating its determinant. The determinant of A is

$$\det(A) = (1)(1)(0) + (0)(1)(1) + (1)(0)(1) - (1)(1)(1) - (0)(0)(1) - (1)(1)(0) = -1,$$

which is non-zero, so A is invertible. Next, we find A^{-1} and then compute $\mathbf{x} = A^{-1}\mathbf{b}$ to express $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (The actual computation of A^{-1} and the resulting expression for $\mathbf{x}$ can be detailed based on standard methods for finding the inverse of a 3×3 matrix.)

This shows that for any vector (a, b, c) in $\mathbb{R}^3$, there exist scalars x_1, x_2, and x_3 such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = (a, b, c)$. Therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

We can use the method above to obtain a general method to find spanning sets.

Theorem 6.2. *Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Then $\mathbb{R}^n = span\{v_1, v_2, \ldots, v_k\}$ if and only if, for the matrix $A = [v_1\ v_2\ \ldots\ v_k]$, the linear system $A\boldsymbol{x} = \boldsymbol{v}$ is consistent for every $\boldsymbol{v} \in \mathbb{R}^n$.*

Proof. In the textbook. □

Example for Theorem with $n = 2$ and $k = 3$

Consider the vector space $\mathbb{R}^2$ and let's choose three vectors v_1, v_2, v_3 in $\mathbb{R}^2$ that are not the standard basis vectors. Specifically, let $v_1 = (1, 1), v_2 = (1, 2), v_3 = (2, 3)$.

Form the matrix $A = [v_1\ v_2\ v_3]$ from these vectors:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Now, consider a general vector $\mathbf{v} = (a, b) \in \mathbb{R}^2$. We want to examine the system $A\mathbf{x} = \mathbf{v}$, where $\mathbf{x}$ is the vector of coefficients (x_1, x_2, x_3) that we are solving for.

The matrix equation becomes:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

After some row operations, the reduced row echelon form of $A^{\#}$ becomes

$$A = \begin{bmatrix} 1 & 0 & 1 & | & 2a - b \\ 0 & 1 & 1 & | & b - a \end{bmatrix}$$

Now, since $rank(A) = rank(A^{\#})$, the system is consistent. Thus, $\mathbb{R}^2 = span\{v_1, v_2, v_4\}$.

Example for Theorem with $n = 3$ and $k = 2$

Consider the vector space $\mathbb{R}^3$ and choose two vectors v_1, v_2 in $\mathbb{R}^3$ that are not the standard basis vectors. Specifically, let $v_1 = (1, 2, 3)$ and $v_2 = (4, 5, 6)$.

Form the matrix $A = [v_1\ v_2]$ from these vectors:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Now, consider a general vector $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$. We want to examine the system $A\mathbf{x} = \mathbf{v}$, where $\mathbf{x}$ is the vector of coefficients (x_1, x_2) that we are solving for.

The matrix equation becomes:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

After some row operations, the augmented matrix form $A^\#$ becomes:

$$A^\# = \begin{bmatrix} 1 & 4 & | & a \\ 0 & 1 & | & b - 2a \\ 0 & 0 & | & c - 3a - (b - 2a) \end{bmatrix}$$

Simplifying the last row gives us $c - b - a = 0$.

Now, the rank of matrix A is 2, which is equal to the rank of the augmented matrix $A^\#$ if and only if $c - b - a = 0$. However, this condition does not hold for every $\mathbf{v} \in \mathbb{R}^3$, indicating that the system $A\mathbf{x} = \mathbf{v}$ may not be consistent for every vector $\mathbf{v}$ in $\mathbb{R}^3$.

Thus, we conclude that $\mathbb{R}^3 \neq \text{span}\{v_1, v_2\}$, as v_1 and v_2 cannot span the entire space of $\mathbb{R}^3$ due to the inconsistency of the system for certain vectors $\mathbf{v}$. This illustrates that two vectors are not sufficient to span $\mathbb{R}^3$.

Example for Theorem with $n = k = 4$

Consider the vector space $\mathbb{R}^4$ and choose four vectors in $\mathbb{R}^4$ as follows: $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 3, 4)$, $v_3 = (4, 3, 2, 1)$, $v_4 = (0, 1, 0, 1)$.

Form the matrix $A = [v_1\ v_2\ v_3\ v_4]$ from these vectors:

$$A = \begin{bmatrix} 1 & 1 & 4 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 1 & 1 \end{bmatrix}$$

Now, consider a general vector $\mathbf{v} = (a, b, c, d) \in \mathbb{R}^4$. The system $A\mathbf{x} = \mathbf{v}$, where $\mathbf{x}$ is the vector of coefficients (x_1, x_2, x_3, x_4), can be written as:

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

After some row operations, the augmented matrix form $A^\#$ becomes:

$$A^\# = \begin{bmatrix} 1 & 1 & 4 & 0 & | & a \\ 0 & 1 & -1 & 1 & | & b - a \\ 0 & 0 & 0 & 2 & | & a - 2b + c \\ 0 & 0 & 0 & 0 & | & a - b - c + d \end{bmatrix}$$

Now, the rank of matrix A is 3, which is equal to the rank of the augmented matrix $A^{\#}$ if and only if $a - b - c + d = 0$. However, this condition does not hold for every $\mathbf{v} \in \mathbb{R}^4$, indicating that the system $A\mathbf{x} = \mathbf{v}$ may not be consistent for every vector $\mathbf{v}$ in $\mathbb{R}^4$.

Thus, we conclude that $\mathbb{R}^4 \neq \text{span}\{v_1, v_2, v_3, v_4\}$, as they cannot span the entire space of $\mathbb{R}^4$ due to the inconsistency of the system for certain vectors $\mathbf{v}$. This illustrates that two vectors are not sufficient to span $\mathbb{R}^4$.