Linear Algebra & Differential Equations

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9 Week 9

9.1 Invertible Matrix Theorem (V. 2)

Building on the introduced concepts of vector spaces, we can expand the list of characteristics that describe invertible matrices by adding new statements.

Theorem 9.1 (Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. The following conditions on A are equivalent:*

1. *A is invertible.*
2. *The equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for every $\boldsymbol{b}$ in $\mathbb{R}^n$.*
3. *The equation $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$.*
4. $rank(A) = n$.
5. *A can be expressed as a product of elementary matrices.*
6. *A is row-equivalent to I_n.*
7. *det(A) is nonzero.*
8. *nullity$(A) = 0$.*
9. *nullspace$(A) = \{\mathbf{0}\}$.*
10. *The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$.*
11. *colspace$(A) = \mathbb{R}^n$, that is, the columns of A span $\mathbb{R}^n$.*
12. *The columns of A form a basis for $\mathbb{R}^n$.*
13. *The rows of A form a linearly independent set of vectors in $\mathbb{R}^n$.*
14. *rowspace$(A) = \mathbb{R}^n$, that is, the rows of A span $\mathbb{R}^n$.*
15. *The rows of A form a basis for $\mathbb{R}^n$.*
16. *A^T is invertible.*

Proofs for the new equivalences will not be provided, as they have been developed from relations explored in prior topics and examples. Our focus will be on the application of the Invertible Matrix Theorem (IMT).

As we cover new concepts in upcoming lectures, new ones will be added to this list. The theorem unites various seemingly unrelated properties and concepts within linear algebra, demonstrating that they are, in fact, different manifestations of the same underlying principle. This helps in understanding the interconnectedness of concepts such as matrix invertibility, system solvability, linear independence, and determinants.

9.2 Linear Transformations

Linear transformations are fundamental operations in linear algebra that map vectors from one vector space to another, preserving the operations of vector addition and scalar multiplication. Intuitively, you can think of them as processes that transform vectors in a way that maintains the "linearity" of the space—straight lines remain straight, and the origin remains fixed. This means that the transformation of a sum of vectors is the same as the sum of their transformations, and similarly for scalar multiples of vectors. These properties make linear transformations versatile, serving as the mathematical foundation for understanding rotations, scaling, shearing, and reflections in geometry, as well as more abstract concepts in areas like computer graphics, quantum mechanics, and machine learning. Essentially, they provide a structured framework for analyzing how different spaces relate to each other and how data within those spaces can be manipulated or interpreted.

Before we delve into defining linear transformations, it's essential to revisit the concept of a function, with our discussion specifically focused on vector spaces.

Definition 9.1. *A function T from a vector space V to a vector space W is defined by a rule that assigns to each element $\mathbf{v}$ in V exactly one element $\mathbf{w}$ in W. Here, V is called the* ***domain*** *of T, and W is the* ***codomain****. The notation $T : V \to W$ signifies that T maps elements from V to W, and for each $\mathbf{v} \in V$, there exists a unique $\mathbf{w} \in W$ such that $T(\mathbf{v}) = \mathbf{w}$.*

Examples.

- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(\mathbf{x}) = 2\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^2$. This function scales every vector in $\mathbb{R}^2$ by a factor of 2.
- Let $k : \mathbb{R} \to \mathbb{R}$ be defined by $k(x) = x^2$, where $x \in \mathbb{R}$.
- Let $T : M_n(\mathbb{R}) \to \mathbb{R}$ be defined by $T(A) = \det(A)$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) =$ some number greater than x. This cannot be a function because we have, for example, $f(2) = 3$ and $f(2) = 4$.

We continue with a definition of a linear transformation between vector spaces.

Definition 9.2. *Let V and W be vector spaces over the same field F. A function $T : V \to W$ is called a* ***linear transformation*** *if the following properties hold for all vectors $u, v \in V$ and any scalar $a \in F$:*

1. $T(u + v) = T(u) + T(v)$
2. $T(a \cdot v) = a \cdot T(v)$

Examples.

1. Consider the scaling transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = 3\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. To prove T is linear, we must show it preserves addition and scalar multiplication. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and let $c \in \mathbb{R}$.

   $T(\mathbf{u} + \mathbf{v}) = 3(\mathbf{u} + \mathbf{v}) = 3\mathbf{u} + 3\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$.

   $T(c\mathbf{u}) = 3(c\mathbf{u}) = c(3\mathbf{u}) = cT(\mathbf{u})$.

   Since T satisfies both conditions, it is a linear transformation.

2. The trace of a square matrix A, denoted as $\text{tr}(A) : M_n(\mathbb{R}) \to \mathbb{R}$, is defined as the sum of its diagonal elements:

   $$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

   where $A = [a_{ij}]$ is an $n \times n$ matrix.

   Given two $n \times n$ matrices A and B, the trace of their sum is:

   $$\text{tr}(A + B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{tr}(A) + \text{tr}(B)$$

   For any scalar c and an $n \times n$ matrix A, the trace of the scalar multiple of A is:

   $$\text{tr}(cA) = \sum_{i=1}^{n} (c \cdot a_{ii}) = c \cdot \sum_{i=1}^{n} a_{ii} = c \cdot \text{tr}(A)$$

   Hence, the trace function satisfies both properties of a linear transformation, making it a linear transformation.

3. Let $S : P_2(\mathbb{R}) \to \mathbb{R}^2$ be a function defined by $S(p(x)) = (p(2), p(4))$ for $p(x) \in P_2(\mathbb{R})$. We will prove that S is a linear transformation by verifying the following two properties for all $p(x), q(x) \in P_2(\mathbb{R})$ and any scalar $c \in \mathbb{R}$:

   $$\begin{aligned} S(p(x) + q(x)) &= ((p + q)(2), (p + q)(4)) \\ &= (p(2) + q(2), p(4) + q(4)) \\ &= (p(2), p(4)) + (q(2), q(4)) \\ &= S(p(x)) + S(q(x)) \end{aligned}$$

$$\begin{aligned} S(c \cdot p(x)) &= ((cp)(2), (cp)(4)) \\ &= (c \cdot p(2), c \cdot p(4)) \\ &= c \cdot (p(2), p(4)) \\ &= c \cdot S(p(x)) \end{aligned}$$

Since S satisfies both necessary properties, S is a linear transformation.

Nonexamples.

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Although f maps real numbers to real numbers, it is not linear because it does not preserve addition or scalar multiplication. For example $f(1) = 1$ and $f(-1) = 1$, so $f(1) + f(-1) = 2$, but $f(1 + (-1)) = f(0) = 0$.

2. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T((x_1, x_2)) = (\cos(x_1), x_2^2)$. We claim that T is not a linear transformation. To demonstrate this, we need to show that T fails to satisfy at least one of the two properties of linear transformations.

$$\begin{aligned} T(2(1,0)) &= T((2,0)) \\ &= (\cos(2), 0^2) \\ &= (\cos(2), 0) \end{aligned}$$

$$\begin{aligned} 2T(1,0) &= 2(\cos(1), 0) \\ &= (2\cos(1), 2 \cdot 0) \\ &= (2\cos(1), 0) \end{aligned}$$

Since $\cos(2) \neq 2\cos(1)$, it follows that:

$$T(2(1,0)) = (\cos(2), 0) \neq (2\cos(1), 0) = 2T(1,0)$$

This calculation shows that the function T does not preserve the scalar multiplication, confirming that T is indeed not a linear transformation.

3. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (a, b, 1)$. To test if T is a linear transformation, let's check the scalar multiplication.

For any scalar $c \in \mathbb{R}$ and vector $(a, b) \in \mathbb{R}^2$, we want

$$T(c(a, b)) = cT(a, b)$$

Computing both sides of the equation gives:

$$T(c(a, b)) = T(ca, cb) = (ca, cb, 1)$$

And

$$cT(a,b) = c(a,b,1) = (ca,cb,c)$$

Clearly,

$$(ca,cb,1) \neq (ca,cb,c)$$

unless $c = 1$. However, for a linear transformation, this equality must hold for all scalars c, including $c \neq 1$. Thus, T is not a linear transformation.

Remark. When you work on different vector spaces, you should be careful about the notation. For example, if you have $T : M_2(\mathbb{R}) \to \mathbb{R}^3$, the inputs are 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and the outputs are triples (x, y, z). As another example, if you have, let say, $S : Fun(\mathbb{R}, \mathbb{R}) \to P_3(\mathbb{R})$, then the inputs are arbitrary functions $f(x)$, but the outputs are polynomials $p(x) = ax^3 + bx^2 + cx + d$. Before determining whether you have a linear transformation or not, first check the domain and range, and write the input and output terms accordingly.

We have the following properties of linear transformations.

Theorem 9.2. *Let V and W be vector spaces over the same field F, and let $T : V \to W$ be a function. Then T is a linear transformation if and only if*

$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$$

for all $v_1, v_2 \in V$ and $c_1, c_2 \in F$.

Proof. ($\Rightarrow$) Suppose T is a linear transformation. Then by the definition of linearity, T must satisfy two properties for any $v, u \in V$ and any scalar $a \in F$:

1. $T(u + v) = T(u) + T(v)$
2. $T(a \cdot v) = a \cdot T(v)$

Consider two vectors $v_1, v_2 \in V$ and scalars $c_1, c_2 \in F$. By applying the definition of a linear transformation, we have:

$$T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) = c_1T(v_1) + c_2T(v_2)$$

This shows that if T is a linear transformation, then $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$.

($\Leftarrow$) Conversely, assume that $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ for all $v_1, v_2 \in V$ and $c_1, c_2 \in F$. To show that T is linear, we need to verify the following properties.

1) Let $v_1, v_2 \in V$. Setting $c_1 = c_2 = 1$, we get

$$T(v_1 + v_2) = T(1 \cdot v_1 + 1 \cdot v_2) = 1T(v_1) + 1T(v_2) = T(v_1) + T(v_2).$$

2) Let $v \in V$ and $c \in F$. Setting $v_2 = 0$ (the zero vector in V) and $c_2 = 0$, we have $T(c \cdot v + 0 \cdot 0) = cT(v) + 0T(0) = cT(v)$, since $T(0) = 0$ in W.

Hence, T is a linear transformation.

□

Proposition 9.3. *Let V and W be vector spaces over the same field, and let $T : V \to W$ be a linear transformation. Then:*

1. $T(\mathbf{0}_V) = \mathbf{0}_W$.
2. $T(-v) = -T(v)$ *for any* $v \in V$.

Proof. 1. Let 0_V be the zero vector in V. Since T is linear, for any scalar c, we have $T(c \cdot 0_V) = c \cdot T(0_V)$. Choosing $c = 0$, we get:

$$T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W$$

where 0_W is the zero vector in W. Since $0 \cdot 0_V = 0_V$, it follows that:

$$T(0_V) = 0_W$$

Thus, T maps the zero vector of V to the zero vector of W.

2. Let v be any vector in V, and let $-v$ be its additive inverse, so $v + (-v) = 0_V$. Applying T to both sides of the equation and using the linearity (additivity) of T, we get:

$$T(v + (-v)) = T(0_V) = 0_W$$

Using the additivity property of T, we have:

$$T(v) + T(-v) = 0_W$$

This shows that $T(-v)$ is the additive inverse of $T(v)$ in W, namely, $T(-v) = -T(v)$. □

Remark. Theorem 9.2 is simpler way to show if T is a linear transformation. Thus, you can use either the definition (show T preserves addition and scalar multiplication) or the shorter way (show the equation in Theorem 9.2).

On the other hand, Proposition 9.3 is useful if you have non-linear transformation. Indeed, if T does not preserve the zero vector or the additive inverses, you can directly say that T is not a linear transformation.

Examples.

1. The function $T : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(ax^2+bx+c) = a$ is a linear transformation.

   To prove that T is a linear transformation, we must show that for any polynomials $p_1(x) = a_1x^2 + b_1x + c_1$ and $p_2(x) = a_2x^2 + b_2x + c_2$ in $P_2(\mathbb{R})$, and for any scalars $d_1, d_2 \in \mathbb{R}$, the following condition holds:

   $$T(d_1p_1(x) + d_2p_2(x)) = d_1T(p_1(x)) + d_2T(p_2(x))$$

   Consider $p_1(x) = a_1x^2 + b_1x + c_1$ and $p_2(x) = a_2x^2 + b_2x + c_2$. Then,

   $d_1p_1(x)+d_2p_2(x) = d_1(a_1x^2+b_1x+c_1)+d_2(a_2x^2+b_2x+c_2) = (d_1a_1+d_2a_2)x^2+(\text{other terms})$

   Applying T to this sum, we get:

   $$T((d_1a_1 + d_2a_2)x^2 + \text{other terms}) = d_1a_1 + d_2a_2$$

   On the other hand,

   $$d_1T(p_1(x)) + d_2T(p_2(x)) = d_1a_1 + d_2a_2$$

   Therefore,

   $$T(d_1p_1(x) + d_2p_2(x)) = d_1T(p_1(x)) + d_2T(p_2(x))$$

   This proves that T satisfies the linearity condition for all polynomials $p_1(x), p_2(x) \in P_2(\mathbb{R})$ and scalars $d_1, d_2 \in \mathbb{R}$, and hence T is a linear transformation.

2. Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ matrices with real entries, and let B be a fixed matrix in $M_n(\mathbb{R})$. The function $S : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $S(A) = AB - BA$ is a linear transformation.

   To prove that S is a linear transformation, we need to show that for any matrices $A_1, A_2 \in M_n(\mathbb{R})$ and any scalars $c_1, c_2 \in \mathbb{R}$, the following property holds:

   $$S(c_1A_1 + c_2A_2) = c_1S(A_1) + c_2S(A_2)$$

   Start by evaluating the left side of the equation:

   $$\begin{aligned} S(c_1A_1 + c_2A_2) &= (c_1A_1 + c_2A_2)B - B(c_1A_1 + c_2A_2) \\ &= c_1A_1B + c_2A_2B - c_1BA_1 - c_2BA_2 \\ &= c_1(A_1B - BA_1) + c_2(A_2B - BA_2) \\ &= c_1S(A_1) + c_2S(A_2) \end{aligned}$$

   This calculation demonstrates that S satisfies the linearity condition, showing that $S(c_1A_1+c_2A_2) = c_1S(A_1)+c_2S(A_2)$ for any $A_1, A_2 \in M_n(\mathbb{R})$ and any scalars $c_1, c_2 \in \mathbb{R}$.

3. The function $U : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $U(x, y, z) = (x + 3y + z, x - y)$ is a linear transformation.

   To prove U is linear, we need to show that for any vectors $\mathbf{v}_1 = (x_1, y_1, z_1)$, $\mathbf{v}_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ and scalars $c_1, c_2 \in \mathbb{R}$,

   $$U(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1U(\mathbf{v}_1) + c_2U(\mathbf{v}_2).$$

First, compute the left-hand side (LHS):

$$\begin{aligned} U(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= U(c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2)) \\ &= U((c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, c_1z_1 + c_2z_2)) \\ &= (c_1x_1 + c_2x_2 + 3(c_1y_1 + c_2y_2) + (c_1z_1 + c_2z_2), \\ &\quad c_1x_1 + c_2x_2 - (c_1y_1 + c_2y_2)) \\ &= (c_1(x_1 + 3y_1 + z_1) + c_2(x_2 + 3y_2 + z_2), \\ &\quad c_1(x_1 - y_1) + c_2(x_2 - y_2)). \end{aligned}$$

Now, compute the right-hand side (RHS):

$$\begin{aligned} c_1U(\mathbf{v}_1) + c_2U(\mathbf{v}_2) &= c_1U(x_1, y_1, z_1) + c_2U(x_2, y_2, z_2) \\ &= c_1(x_1 + 3y_1 + z_1, x_1 - y_1) + c_2(x_2 + 3y_2 + z_2, x_2 - y_2) \\ &= (c_1(x_1 + 3y_1 + z_1) + c_2(x_2 + 3y_2 + z_2), \\ &\quad c_1(x_1 - y_1) + c_2(x_2 - y_2)). \end{aligned}$$

Since the LHS and RHS are equal,

$$U(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1U(\mathbf{v}_1) + c_2U(\mathbf{v}_2),$$

it follows that U satisfies the condition for a linear transformation. Therefore, U is indeed a linear transformation.

Nonexamples.

1. The function $T : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$, where A is a 2×2 matrix, is not a linear transformation.

   Consider the 2×2 identity matrix I_2:

   $$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   The determinant of I_2 is $T(I_2) = \det(I_2) = 1$.

   Now, consider the matrix $-I_2$, which is the matrix I_2 multiplied by -1:

   $$-I_2 = -1 \cdot I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   The determinant of $-I_2$ is $T(-I_2) = \det(-I_2) = (-1)(-1) - (0)(0) = 1$. According to the preservation of additive inverses, we would expect that $T(-I_2) = -T(I_2)$. However, we have:

   $$T(-I_2) = 1 \neq -1 = -T(I_2)$$

   Therefore, T is not a linear transformation.

2. The function $S : P_1(\mathbb{R}) \to \mathbb{R}$, defined by $S(ax + b) = b + 1$ for any polynomial $ax + b \in P_1(\mathbb{R})$, where $a, b \in \mathbb{R}$, is not a linear transformation.

   Evaluating S at the zero polynomial, we have:

$$S(0x + 0) = S(0) = 0 + 1 = 1$$

   However, for S to be a linear transformation, we require $S(0) = 0$. Since $S(0) = 1 \neq 0$, this demonstrates that S is not a linear transformation.

3. The function $T : \mathbb{R}^2 \to M_2(\mathbb{R})$ defined by $T(a, b) = \begin{pmatrix} a^2 & b^2 \\ 0 & 1 \end{pmatrix}$ for any $(a, b) \in \mathbb{R}^2$ is not a linear transformation.

   Consider the zero vector in $\mathbb{R}^2$, which is $0 = (0, 0)$. Applying the function T to this vector yields:

$$T(0, 0) = T(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

   However, the zero matrix in $M_2(\mathbb{R})$ is:

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   Clearly, $T(0, 0) \neq 0$ because $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   This shows that T does not satisfy the property $T(0) = 0$ for linear transformations. Therefore, T is not a linear transformation.

9.3 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are crucial in both the study and application of linear algebra. The following is the main example of such transformations.

Let A be an $m \times n$ matrix, then $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. Indeed, we have

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y}),$$

$$T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x}).$$

This linear transformation is called **matrix transformation**.

Example. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 5 \\ -1 & 3 \end{bmatrix}$, then $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by $T(\mathbf{x}) = A\mathbf{x}$ is a transformation, we can expand it as

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 3x + 4y \\ 5y \\ -x + 3y \end{bmatrix}.$$

Thus, we can also write the transformation as

$$T((x, y)) = (2x + y, 3x + 4, 5y, -x + 3y).$$

The distinctive feature of the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ is that any such transformation is a matrix transformation. We have the following theorem.

Theorem 9.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is described by the matrix transformation*

$$T(\mathbf{x}) = A\mathbf{x}$$

where A is the $m \times n$ matrix

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \dots\ T(\mathbf{e}_n)]$$

and $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the standard basis vector in $\mathbb{R}^n$.

Proof. Consider any vector $\mathbf{x} \in \mathbb{R}^n$ which can be expressed as a linear combination of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. That is,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n,$$

where $x_1, x_2, \dots, x_n$ are the components of $\mathbf{x}$.

Since T is linear, by the properties of linearity, we have

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n). \end{aligned}$$

Each $T(\mathbf{e}_i)$ is a vector in $\mathbb{R}^m$, and these vectors can be used as columns to form the $m \times n$ matrix A. Hence, the equation above represents the matrix-vector product of A and $\mathbf{x}$, where

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \dots\ T(\mathbf{e}_n)].$$

Therefore, for any vector $\mathbf{x} \in \mathbb{R}^n$, the image of $\mathbf{x}$ under the transformation T can be computed as the matrix-vector product $A\mathbf{x}$, proving that T is indeed represented by the matrix A. □

Examples.

1. For $T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 5x_2)$,
$$T(\mathbf{e}_1) = T(1,0) = (3 \cdot 1 - 2 \cdot 0, 1 \cdot 1 + 5 \cdot 0) = (3, 1),$$
$$T(\mathbf{e}_2) = T(0,1) = (3 \cdot 0 - 2 \cdot 1, 1 \cdot 0 + 5 \cdot 1) = (-2, 5).$$
Thus, the matrix A is:
$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}.$$

2. For $T(x_1, x_2) = (x_1 + 3x_2, 2x_1 - 7x_2, x_1)$,
$$T(\mathbf{e}_1) = T(1,0) = (1 + 3 \cdot 0, 2 \cdot 1 - 7 \cdot 0, 1) = (1, 2, 1),$$
$$T(\mathbf{e}_2) = T(0,1) = (0 + 3 \cdot 1, 2 \cdot 0 - 7 \cdot 1, 0) = (3, -7, 0).$$
The matrix A is:
$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -7 \\ 1 & 0 \end{bmatrix}.$$

3. For $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_3 - x_1)$,
$$T(\mathbf{e}_1) = T(1,0,0) = (1 - 0 + 0, 0 - 1) = (1, -1),$$
$$T(\mathbf{e}_2) = T(0,1,0) = (0 - 1 + 0, 0 - 0) = (-1, 0),$$
$$T(\mathbf{e}_3) = T(0,0,1) = (0 - 0 + 1, 1 - 0) = (1, 1).$$
The matrix A is:
$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

4. For $T(x_1, x_2, x_3) = x_1 + 5x_2 - 3x_3$,
$$T(\mathbf{e}_1) = T(1,0,0) = 1 \cdot 1 + 5 \cdot 0 - 3 \cdot 0 = 1,$$
$$T(\mathbf{e}_2) = T(0,1,0) = 1 \cdot 0 + 5 \cdot 1 - 3 \cdot 0 = 5,$$
$$T(\mathbf{e}_3) = T(0,0,1) = 1 \cdot 0 + 5 \cdot 0 - 3 \cdot 1 = -3.$$
The matrix A is:
$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 5 & -3 \end{bmatrix}.$$

5. For $T(x_1, x_2, x_3) = (x_3 - x_1, -x_1, 3x_1 + 2x_3, 0)$,
$$T(\mathbf{e}_1) = T(1,0,0) = (0 - 1, -1, 3 \cdot 1 + 2 \cdot 0, 0) = (-1, -1, 3, 0),$$
$$T(\mathbf{e}_2) = T(0,1,0) = (0 - 0, 0, 3 \cdot 0 + 2 \cdot 0, 0) = (0, 0, 0, 0),$$
$$T(\mathbf{e}_3) = T(0,0,1) = (1 - 0, 0, 3 \cdot 0 + 2 \cdot 1, 0) = (1, 0, 2, 0).$$
The matrix A is:
$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$