Practice for midterm 1

True — False A is a 3×3 matrix such that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then A does not have an inverse.

True — False If A and B are diagonal $n \times n$ matrices, then $AB = BA$.

True — False If A and B are $n \times n$ matrices, and A is invertible, then AB is invertible.

True — False If reduced row-echelon form of a matrix A is the zero matrix, then A must be the zero matrix.

True — False Suppose A is a 3×5 matrix. Then $rank(A) = 3$.

True — False Suppose A is a 3×5 matrix. Then $A\mathbf{x} = \mathbf{b}$ has a unique solution.

True — False If an $m \times n$ system is inconsistent, then the reduced row-echelon form of the augmented matrix must have $n + 1$ nonzero rows.

True — False If a square matrix A can be reduced to the identity matrix by row operations, then A is invertible.

True — False A linear system with less equations than variables cannot have a solution.

True — False A homogeneous system of equations can be inconsistent.

True — False If each element of an $n \times n$ matrix is doubled, then the determinant of the matrix also doubled.

True — False If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$.

True — False For every positive integer n, $\text{adj}(I_n) = I_n$.

True — False If A is $n \times n$ matrix and c is a scalar, then $\text{adj}(cA) = c(\text{adj}(A))$.

True — False The trace of a matrix is the product of the elements on the main agonal.

True — False If A and B are matrix functions such that $A(0) = B(0)$, then A and B are the same matrices.

True — False If A and B are symmetric $n \times n$ matrices, then $A + B$ is also symmetric.

True — False For any matrices A and B of same dimensions, we have

$$rank(A + B) = rank(A) + rank(B).$$

True — False If A is a 5×5 matrix of rank 4, then A is not invertible.

True — False Every matrix can be expressed as a product of elementary matrices.

1. Consider the system of equations $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 6 \\ 7 & 10 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

   (a) Write the augmented matrix.

   (b) Reduce it into row-echelon form.

   (c) Determine for which $\mathbf{b}$, the system has no solutions.

2. Write down the set of all solutions of the following system of linear equations

$$\begin{aligned} x - y + z &= 1 \\ 2x - y + z &= 1 \\ 3x - 2y + 2z &= 2. \end{aligned}$$

3. Prove that if A and B are $n \times n$ diagonal matrices, then AB is also diagonal matrix.

4. Suppose A and B are 2×2 matrices such that $\det(B) = 8$ and $\det(A^3) = \det(B^2)$. Determine the value of

$$\det(3A^T B A^{-1} B^{-1} A).$$

5. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 4 & 2 \\ 2 & 2 & 10 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 5 \end{bmatrix}$. Compute **only** the entry at the third row and the second column of $\operatorname{adj}(A)$.

6. Consider the matrices $A = \begin{bmatrix} a & b & c \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} x & y & z \\ -3 + bx & 7 + by & 2 + bz \\ a & b & c \end{bmatrix}$. Suppose $\det(A) = 3$. Find $\det(2B)$.

7. Consider a linear system whose augmented matrix is of the form

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & a & a-3 \\ 0 & 0 & a-4 & a-3 \end{array}\right].$$

   (a) For which values of a, the system has no solution?

   (b) For which values of a, the system has a unique solution?

   (c) For which values of a, the system has infinitely many solution?

8. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$.

   (a) Find $\det(A)$.

   (b) Compute AA^T.

   (c) Determine A^{-1} if it exists.

9. A square matrix A is called *idempotent* if $A^2 = A$. Prove that if A is idempotent and invertible, then A must be the identity matrix.

10. Assume $rank(A) = 2$ for the matrix $A = \begin{bmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{bmatrix}$. Find the value of a.

11. Use Cramer's rule to solve

$$\begin{aligned} 2x &= 6 \\ 2x + 2y + z &= 5 \\ 4x + y + z &= 11. \end{aligned}$$

12. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & a & 2 \end{bmatrix}$.

    (a) Find $\det(A)$.

    (b) For which values of a, A is invertible?

    (c) Use Gauss-Jordan method to find the inverse of A when $a = -1$.